Applications of Orlicz-Lorentz Spaces in Gateaux Differentiability

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Abstract
The study aims to study Gateaux differentiability of the functional $\varphi(\omega, \emptyset)(f) = \int_0^{\infty} . \emptyset(f^*)\omega$ and of the Luxemburg norm, it follows the descriptive method and the study found that we can obtain the one-sided Gateaux derivatives in both cases by characterizing those points where the Gateaux derivative of the norm exists,we obtain a characterization of best $\varphi_{\omega,\emptyset}$-approximants from convex closed subsets, there a relation between best $\varphi_{\omega,\emptyset}$-approximants and best approximants from a convex set.
Keywords: Gateaux derivatives, Orlicz-Lorentz space, best approximants.

المستخلص
هدفت هذه الدراسة إلى دراسة مشتقة جيتوكس للدالى $\varphi._{\omega,\emptyset}(f) = \int_0^{\infty} . \emptyset(f^*)$ ولنظيم لكسمبورج و إتبعت الدراسة المنهج الوصفى وتوصلت إلى إن بالإمكان الحصول على مشتقة جيتوكس ذات الجانب الواحد فى كل الأحوال بتوصيف هذه النقاط التى تمثل نظيم خروج ،وتحصلنا على توصيف للتقريب الأفضل من مجموعات جزئية محدبة مغلقة ، هناك علاقة بين التقريب الأفضل ل$\varphi_{\omega,\emptyset}$ و التقريب الافضل من مجموعة محدبة.
كلمات مفتاحية: مشتقة جيتوكس ، فضاء أورليش- لورنتز، أفضل تقريب

1.Introduction

Let M_0 be the class of all real extended φ-measurable functions on$[0, \alpha)$,$0 < \alpha \leq \infty$, where is the Lebesgue measure. As usual, for $f \in M_0$we denote its distribution function by $._{f(\Omega)} = \mu(\{0 \leq x < \alpha: |f(x)| > \Omega\})(\Omega \geq 0)$, and its decreasing rearrangement by
$f^*(1+2\epsilon) = inf\ \{\Omega: \mu_f(\lambda) \leq 1+2\epsilon\}\ \left(\epsilon \geq \frac{-1}{2}\right)$.
If two functions f and g have the same distribution functions we say they are equimeasurable and we denote it by $f \sim g$.
For other properties of μ_f and f^*, (see [1] Bennet and Sharpley, 1988,pp.36-42).
Let $\emptyset: R_+ \to R_+$ be differentiable, convex , $\emptyset(0) = 0, \emptyset(1+2\epsilon) > 0\ \ for\ \epsilon > \frac{-1}{2}$ and let
$\omega : (0, \alpha) \to (0, \infty)$ be a weight function, non-increasing and locally integrable.
If $\alpha = \infty$,we assume $\omega(1+2\epsilon) = 0$ and $\int_0^{\infty} . w(1+2\epsilon) d\mu(1+2\epsilon) = \infty$.
For $f \in M_0$ let $\varphi_{\omega,\emptyset}(f) = \int_0^{\alpha} \varphi\, \emptyset\big(f^*(1+2\epsilon)\big)\omega(1+2\epsilon)\, d\mu(1+2\epsilon)$
In ([4] Hudzik, Kamińska an Mastylo,2002 and [6] Kamińska,1990 -[8] Kamińska,1991), several authors studied geometric properties of the regular Orlicz-Lorentz spaces,$\big\{f \in M_0: \varphi_{\omega,\emptyset}(\lambda f) < \infty\ for\ some\ \Omega > 0\big\}$.The main ojective of this section is to study differentiability properties in the following subspace

$$\Lambda_{\omega,\emptyset} := \big\{f \in M_0: \varphi_{\omega,\emptyset}(\Omega f) < \infty\ for\ \ all\ \ \Omega > 0\big\},$$

which appears to be convenient for our purpose. Under the norm given by

$$\|f\|_{\omega,\emptyset} = inf\ inf\ \left\{\epsilon > 0: \varphi_{\omega,\emptyset}\left(\frac{f}{\epsilon}\right) \leq 1\right\},$$

$.\varphi_{\omega,.\emptyset}$ is a Banach space ([6] Kamińska,1990).It is clear that if ω is constant, $\Lambda_{\omega,\emptyset}$ becomes a subspace of finite elements $L_{\emptyset}^{\infty}$ of the Orlicz space $L_{\emptyset}$ (see [15] Rao and Ren, 1991). On the other hand setting $\emptyset(1+2\epsilon) = (1+2\epsilon)^{1+\epsilon}, 0 \leq \epsilon < \infty$, we obtain the Lorentz space $L_{(\omega,1+\epsilon)}$and
$\varphi_{\omega,\emptyset}(f) = \|f\|_{(\omega,1+\epsilon)}^{1+\epsilon}$. These weighted Lorentz spaces as a generalization of Lorentz space have been studied in ([3] Halperin, 1953). If $\omega(1+2\epsilon) = \left(\frac{1+\epsilon}{1-\epsilon}\right)(1+2\epsilon)^{\left(\frac{1-\epsilon}{1+\epsilon}\right)-1}, 0 \leq -\epsilon \leq \epsilon < \infty$, a good reference for a description of these spaces in $L(1+\epsilon, 1-\epsilon)$ spaces in ([5] Hunt, 1966).A function $\sigma: [0,\ \alpha) \to [0,\ \alpha)$is called a measure preserving transformation (m.p.t) if for each μ-measurable set $I \subset [0,\ \alpha), \sigma^{-1}(I)\ is\ \mu$-measurable and $\big(\sigma^{-1}(I)\big) = \mu(I)$. It is very important to emphasize that any m.p.t induces equimeasurability, that is, if
$g \in M_0$then$|g| \circ \sigma$ is a μ-measurable function on$[0, \alpha)$and $|g| \circ \sigma \sim |g|$.For $g \in M_0$,we denote $(g) := \{0 \leq x < \alpha: g(x) \neq 0\}$ .In view of the assumptions on the weight ω, if
$f \in \Lambda_{\omega,\emptyset}$, then $f^*(1+2\epsilon) = 0$.In consequence,by Ryff's Theorem (see [1] Bennet and Sharpley, 1988) there is an m.p.t $\sigma: (f) \to (f^*)$ such that

$$|f| = f^* \circ \sigma \quad \mu - a.e.\ on\ (f)(1.1)$$

Moreover, if $\sigma: (f) \to (f^*)$ is any m.p.t fulfilling (1.1), then

$$\varphi_{\omega,\emptyset}(f) = \int_{(f)}^{\infty} \varphi_{\omega,\emptyset}..\emptyset(|f|)d\mu.$$

In fact, since $\emptyset(f^*)w \sim \emptyset(|f|)w(\sigma)$, so their integrals are equal (see [1] Bennet and Sharpley, 1988).Let $T: \Lambda_{\omega,\emptyset} \to R$ be a functional.For $f, h \in \Lambda_{\omega,\emptyset}$,we will use in this work the one-sided Gateaux derivatives $\gamma_T^+(f,h) = \frac{T\left(f+h(1-2\epsilon)\right)-T(f)}{1-2\epsilon}$ and
$\gamma_T^-(f,h) = \frac{T\left(f+h(1-2\epsilon)\right)-T(f)}{1-2\epsilon}$. (see [2] Carothers, Haydon and Lin,1993) showed that if $\alpha = \infty, \emptyset(1+2\epsilon) = (1+2\epsilon)^{1+\epsilon}$, 0< $\epsilon < \infty$, and ω is astrictly decreasing function, then
$.(f,h) = (1+\epsilon)\int_0^{\infty} . \omega\big(\tau_{f,h}\big)|f|^{\epsilon}(1-2\epsilon)g(f)h\, d\mu$, where $\big(\tau_{f,h}\big)$ is defined by

$$(\tau_{f,h})(x) = \mu_f(|f(x)|) + \mu(\{y: |f(y)| = |f(x)| and\ h(y)(1-2\epsilon)g(f(y)) > h(x)\ (1-2\epsilon)g(f(x))\}) + \mu(\{y: |f(y)| = |f(x)|, h(y)(1-2\epsilon)g(f(y)) = h(x)\ (1-2\epsilon)g(f(x)) and\ y \leq x\}). \quad (1.2)$$

It is known that $\tau_{f,h}$is an m.p.t and $|f| = f^* \circ \tau_{f,h}$ μ-a.e. on (f) (see [16] Ryff, 1970).
In one sided Gateaux derivatives in $\Lambda_{\omega,\emptyset}$, we generalize this result .Using atechnique similar to that in ([10] Levis and Cuenya,2004,Theorem 2.6), we compute the one-sided Gateaux derivative of the modular for $0 < \alpha \leq \infty, \omega$ a non-increasing function, and $\emptyset$, a convex function.Here,we need to work with a suitable m.p.t. .Also, we obtain the one-sided Gateaux derivative for the norm $\|\cdot\|_{.\omega,\emptyset}$, called also the Luxemburg norm.
We say that $f \in \Lambda_{\omega,\emptyset}$ is a smooth point for T if there exists the Gateaux derivative of the functional T in f, i.e if $\gamma_T^+(f,h) = \gamma_T^-(f,h)$ for all $h \in \Lambda_{\omega,\emptyset}$and we denote it by $\gamma_T(f,h)$.The set of smooth points for the functional $\varphi_{\omega,\emptyset}$ was investigated in ([10] Levis and Cuenya,2004).
Let $K \subset \Lambda_{\omega,\emptyset}$ and $f \in \Lambda_{\omega,\emptyset}$ be given, and considerthe problem of finding $h^\bullet \in K$ such that

$$T(f-h^\bullet) ==: E_T\,(f,K). \qquad (1.3)$$

Denote by $P_T(f,K)$ the set of all $h^\bullet \in K$ fulfilling (1.3). Each element of $P_T(f,K)$ will be called the best T-approximant of f from K. If T is the Luxemburg norm, we only say the best approximant from K. Let $\alpha = 1$, let $\emptyset(1+2\epsilon) = (1+2\epsilon)^{1+\epsilon}$ with $0 \leq \epsilon < \infty$, let f be a simple function in $\Lambda_{\omega,\emptyset}$, and let $K := \{g \in \Lambda_{\omega,\emptyset} : g\ is\ constant\}$.In ([11] Levis and Cuenya,2004), we give a characterization of the best, $\varphi_{\omega,\emptyset}$-approximants of f from K and we show the way to obtain the best $\varphi_{\omega,\emptyset}$-approximants maximum and minimum, which will be denoted by $\underline{f}$ and f respectively.We give a characterization of the best $\varphi_{\omega,\emptyset}$-approximants of $f \in \Lambda_{\omega,\emptyset}$ from a convex closed set, K, and we establish a relation between the best $\varphi_{\omega,\emptyset}$-approximants and the best approximants from K.Finally, we give a characterization of the best constant $\varphi_{\omega,\emptyset}$-approximants and we calculate the best constant $\varphi_{\omega,\emptyset}$-approximants maximum and minimum)[12] Levis and Cuenya, 2007).

2.One sided Gateaux derivatives in $\Lambda_{\omega,\emptyset}$

We let $f,h \in \Lambda_{\omega,\emptyset}$for each $\epsilon \geq \frac{1}{2}$,we consider any m.p.t.$\sigma_{f+h(1-2\epsilon)}: (f+h(1-2\epsilon)) \to ((f+h(1-2\epsilon))^*)$ such that
$|f+h(1-2\epsilon)| = (f+h(1-2\epsilon))^* \circ \sigma_{f+h(1-2\epsilon)}$-a.e.$(f+h(1-2\epsilon))$.
In ([10] Levis and Cuenya, 2004),we showed that $\sigma_f\,(x) = \sigma_{f+h(1-2\epsilon)}\mu$-a.e on $E(f) \cap (f)$ where $E(f) := \{0 \leq x < \alpha: \mu\{|f| = |f(x)|\} = 0\}$.
However, we give an example which shows that this result does not hold on the whole (f))[12] Levis and Cuenya, 2007).

Example 2.1

For $\alpha = 1$, let $f = 2\chi_{[0,\frac{1}{2})} + \chi_{[\frac{1}{2},1)}$ and $h = \chi_{[\frac{1}{4},\frac{3}{4})}$.For $\frac{1}{2} < \epsilon < 0$,
we consider the m.p.t. defined by

$$\cdot_{(f+h(1-2\epsilon))}(x) = \left(x+\frac{1}{4}\right)\chi_{[0,\frac{1}{4})} + \left(x-\frac{1}{4}\right)\chi_{[\frac{1}{4},\frac{1}{2})} + x\chi_{[\frac{1}{2},1)},$$

and for $\frac{1}{2} < \epsilon < 0$ the m.p.t. defined by

$$\cdot_{(f+h(1-2\epsilon))}(x) = x\chi_{[0,\frac{1}{2})} + \left(x+\frac{1}{4}\right)\chi_{[\frac{1}{2},\frac{3}{4})} + \left(x-\frac{1}{4}\right)\chi_{[\frac{3}{4},1)}.$$

We observe that for all $0 \leq x < 1, \sigma_{(f+h(1-2\epsilon))}(x)$ does not exist.
Now our purpose is to define a sequence of m.p.t.$\sigma_{f+(1-2\epsilon)_n h}$
such that $\sigma_{f+(1-2\epsilon)_n h}\,(x)$exist for $x \in (f) \cup (h)$ .To prove it we need some auxiliary lemmas.

Lemma 2.2

Let $R \subset [0,\alpha)$ be a μ-measurable set with $\mu(R) = b > 0$.
Then $\sigma: [\,0,\alpha) \to [0,b)$defined by $\sigma(x) = \mu(R \cap [0,x])$is a non-decreasing continuous function with $\sigma(0) = 0$ and $\sigma(x) = b$.

Lemma 2.3

Let σ be the function given in Lemma 2.2.Then $\sigma: R \to [0,b)$ is an m.p.t.We denote such σ by σ_R.
Proof. Let $0 \le \Omega < b$. From Lemma 2.2 there exists $0 \le x < \alpha$ such that $\sigma(\,x) = \Omega$.
We consider $x_\Omega =sup\ sup\ \{x: \sigma(x) = \Omega\}$ . .Since , $x_\Omega\ < \alpha$
show that $\{x \in R: \sigma_R(x) > \Omega\} = R \cap (x_\Omega, \alpha)$.
Let $x \in R$ be such that $\sigma_R(x) > \Omega$. $If\ x \le x_\Omega$, from Lemma 2.2 $\sigma_R(x) \le \sigma_R(x_\Omega) = \Omega$ and this a contradiction. Thus $x \in R \cap (x_\Omega, \alpha)$.On the other hand,if

$$x \in R \cap (x_\Omega, \alpha), \sigma_R(x) \ge \sigma_R(x_\Omega) = \Omega \text{ .If } \sigma_R(x) = \Omega$$

then x_Ω is not the supreme and this is another contradiction.Therefore $\sigma_R(x) > \Omega$.Then

$$\mu_{\sigma_R}(\Omega) = \mu\big(R \cap (x_\Omega, \alpha)\big). \qquad (2.1)$$

Now , we consider $g(\,x) = x, 0 \le x < b$.
From (2.1) and the continuity of σ, we have $\mu_{\sigma_R}(\Omega) = b - \sigma(x_\Omega) = b - \Omega = \mu_g(\Omega)$.
In consequence σ_R and g are equimeasurable functions. If I is any μ-measurable subset of $[0,b)$, then $g^{-1}(I) = I$ is a μ-measurable set. From ([1] Bennet and Sharpley, 1988,Lemma 7.3), $\sigma_R^{-1}(I)$ is μ-measurable and $\mu\big(\sigma_R^{-1}(I)\big) = \mu\big(g^{-1}(I)\big) = \mu(I)$.The proof is complete.
Let $f \in \Lambda_{\omega,\emptyset}$. By redefining f, if necessary, on a set of μ-measure zero , we may assume that $|f|$and f^* have the same non-null range, say $R(\,f)$. For $\Omega \in R(\,f)$, we consider $C_f(\Omega) \coloneqq \{0 \le x < \alpha: |f(x)| = \Omega\}$ and $I_f(\Omega) \coloneqq \left\{\epsilon > \frac{-1}{2}: f^*(1+2\epsilon) = \Omega\right\}$. So,
$\mu\left(C_f(\Omega)\right) = \mu\left(I_f(\Omega)\right) < \infty$. By Lemma 2.3, the function $\sigma_\Omega: C_f(\Omega) \to I_f(\Omega)$
defined by $\sigma_\Omega(x) = \mu_f(\Omega) + \mu\left(C_f(\Omega)\right) \cap [0,x]$ is an m.p.t .Thus, the function

$$\sigma_f(x) = \sigma_\Omega(x)\left(x \in C_f(\Omega)\right) \qquad (2.2)$$

is an m.p.t. from (f) onto (f^*) .

Remark 2.4

Given $f\ \in\ \Lambda_{\omega,\emptyset}$ we can write $\sigma_f(x) = \mu_f(|f(x)|) + \mu(\{y: |f(y)| = |f(x)|\ and\ y \le x\}\,)$. If $\mu\big((f)\,\big) < \infty$, then σ_f is an m.p.t. from $[0,\alpha)$ into $[0,\alpha)$.

Lemma 2.5

Let $f\,,h \in \Lambda_{\omega,\emptyset}$ and let $\Omega > 0$. If $\mu\left(C_f(\Omega)\right) = 0$, then μ_f is acontinuous function at Ω and $\mu_{f+h(1-2\epsilon)}(\Omega)\ = \mu_f(\Omega)$.
Proof. Since μ_f is a right continuous function, it is sufficient to show that
$\mu_f(1-2\epsilon) = \mu_f(\Omega)$. Let$((1-2\epsilon)_n)_n$ be a sequence such that
$0 < (1-2\epsilon)_n \uparrow \Omega$ and $C_n = \{y: |f(y)| > (1-2\epsilon)_n\}$.
Clearly $C_{n+1} \subset C_n$ and $\mu(C_1) < \infty$.As $\mu\left(C_f(\Omega)\right) = 0$, we get

$$\mu_f\ ((1-2\epsilon)_n) = \mu(\cap_{n=1}^{\infty} C_n) = \mu(\{y: |f(y)| \ge \Omega\}) = \mu_f(\Omega).$$

Now we shall prove that
Using properties of the distribution function we obtain

$$\mu_{f+h(1-2\epsilon)}(\Omega) = \mu_{f+h(1-2\epsilon)}\left(\left(1-\sqrt{|1-2\epsilon|}\right)\Omega + \sqrt{|1-2\epsilon|}\Omega\right)$$
$$\le \mu_f\left(\left(1-\sqrt{|1-2\epsilon|}\right)\Omega\right) + \ \mu_h\left(\frac{\sqrt{|1-2\epsilon|}}{|1-2\epsilon|}\Omega\right) (2.3)$$

and $\mu_f(\Omega) \leq \underline{\mu_{f+h(1-2\epsilon)}}(\Omega)$.

Since $h \in \Lambda_{\omega,\emptyset}$ we have $\mu_h\left(\frac{\sqrt{|1-2\epsilon|}}{|1-2\epsilon|}\Omega\right) = 0$.

In addition $\mu_f\left(\left(1-\sqrt{|1-2\epsilon|}\right)\Omega\right) = \mu_f(\Omega)$.

So (2.3) implies that $\underline{\mu_{f+h(1-2\epsilon)}}(\Omega) \leq \mu_f(\Omega)$.The proof is complete.

Remark 2.6

Let $f \in \Lambda_{\omega,\emptyset}$ and $\Omega > 0$. Clearly $\mu\left(\left\{y: \left|f(y)\chi_{\underline{C_f(\Omega)}}(y)\right| = \Omega\right\}\right) = 0$, where $\underline{A} = [0,\alpha) - A$. Thus, Lemma 2.5 implies that $\mu_{f\chi_{\underline{C_f(\Omega)}}}$ is continuous at Ω.

Lemma 2.7

Let $f, h \in \Lambda_{\omega,\emptyset}$ .If $\Omega > 0$ and $x \in C_f(\Omega)$,
then $\mu_{(f+(1-2\epsilon)h)\chi_{\underline{C_f(\Omega)}}}(|f(x) + (1-2\epsilon)h(x)|) = \mu_f(\Omega)$.

Proof. It is enough to operate on decreasing sequence $(1-2\epsilon)_n$, which we denote by $(1-2\epsilon)_n \downarrow 0$ .Since $|f(x) + (1-2\epsilon)_n h(x)| \frac{\sqrt{(1-2\epsilon)_n}}{(1-2\epsilon)_n} = \infty$and $f, h \in \Lambda_{\omega,\emptyset}$, then

$$\left(|f(x) + (1-2\epsilon)_n h(x)| \frac{\sqrt{(1-2\epsilon)_n}}{(1-2\epsilon)_n}\right) = 0 . \qquad (2.4)$$

In fact, (2.4) is obvious if $\alpha < \infty$. For $\alpha = \infty$, (2.4) follows from our assumption $\int_0^\infty w(1+2\epsilon)d\mu(1+2\epsilon) = \infty$. Using properties of the distribution function we obtain,

$$\begin{aligned}\mu_{(f+(1-2\epsilon)_n h)\chi_{\underline{C_f(\Omega)}}}&(|f(x) + (1-2\epsilon)_n h(x)|) \\ &\leq \mu_{f\chi_{\underline{C_f(\Omega)}}}\left(|f(x) + (1-2\epsilon)_n h(x)|\left(1-\sqrt{(1-2\epsilon)_n}\right)\right) \\ &+ \mu_{h\chi_{\underline{C_f(\Omega)}}}\left(|f(x) + (1-2\epsilon)_n h(x)| \frac{\sqrt{(1-2\epsilon)_n}}{(1-2\epsilon)_n}\right).\end{aligned}$$

So, (2.4) implies that

$$\mu_{(f+(1-2\epsilon)_n h)\chi_{\underline{C_f(\Omega)}}}(|f(x) + (1-2\epsilon)_n h(x)|) \leq \mu_f(\Omega). \qquad (2.5)$$

On the other hand ,as shown in ([1] Bennet and Sharpley, 1988), it is known that

$$\mu_f(\Omega) = \mu_{f\chi_{\underline{C_f(\Omega)}}}(\Omega) \leq \mu_{(f+(1-2\epsilon)_n h)\chi_{\underline{C_f(\Omega)}}}(|f(x) + (1-2\epsilon)_n h(x)|). \quad (2.6)$$

From (2.5) and (2.6) the proof follows immediately.

Lemma 2.8

Let $f, h \in \Lambda_{\omega,\emptyset}$. If $\Omega \geq 0$ and $x \in C_f(\Omega)$,
then $\mu\left(\left\{y \in \underline{C_f}(\Omega): |f(y) + (1-2\epsilon)h(y)| = |f(x) + (1-2\epsilon)h(x)| and\ y \leq x\right\}\right) = 0$.

Proof. Let $(1-2\epsilon)_n \downarrow 0, C_n \coloneqq \left\{y \in \underline{C_f(\Omega)}: |f(y) + (1-2\epsilon)_n h(y)| = |f(x) + (1-2\epsilon)_n h(x)|\ and\ y \leq x\right\}$

$and\ D_n \coloneqq \left\{y \in \underline{C_f(\Omega)} : ||f(y)| - f(x)||| \leq (1-2\epsilon)_n(|h(y)| + |h(x)|) and\ y \leq x\right\}$.

Clearly $C_n \subset D_n \subset [0,x], D_{n+1} \subset D_n\ and \bigcap_{n=1}^\infty D_n = \emptyset$.
Then $0 \leq \mu(C_n) \leq \mu(D_n) = \mu(\bigcap_{n=1}^\infty D_n) = 0$.

Lemma 2.9

Let $f, h \in \Lambda_{\omega,\emptyset}$. If $\Omega > 0$ and $x \in C_f(\Omega)$, then

$$\mu(\{y \in C_f(\Omega): |f(y) + (1-2\epsilon)h(y)| > |f(x) + (1-2\epsilon)h(x)|\}) = \mu(\{y \in C_f(\Omega): (1-2\epsilon)g(f(y)h(y) > (1-2\epsilon)g(f(x)h(x)\}).$$

Proof. Let $(1-2\epsilon)_n \downarrow 0$,

$$R_n \coloneqq \left\{y \in C_f(\Omega): |f(y) + (1-2\epsilon)_n h(y)| > |f(x) + (1-2\epsilon)_n h(x)| \; and \; |h(y)| \leq \frac{\Omega}{(1-2\epsilon)_n}\right\},$$

$(1-2\epsilon)_n \coloneqq \left\{y \in C_f(\Omega): |f(y) + (1-2\epsilon)_n h(y)| > |f(x) + (1-2\epsilon)_n h(x)| \; and |h(y)| > \frac{\Omega}{(1-2\epsilon)_n}\right\}$and $R \coloneqq \{y \in C_f(\Omega): (1-2\epsilon)g(f(y)h(y) > (1-2\epsilon)g(f(x)h(x)\}$.

As $\mu((1-2\epsilon)_n) \leq \mu_h\left(\frac{\Omega}{(1-2\epsilon)_n}\right), \mu((1-2\epsilon)_n) = 0$.

Then, it will be sufficient to prove that

$\mu(R_n) = \mu(R)$ Let $N \in N$ be such that if $n \geq N$,

$|f(x) + (1-2\epsilon)_n h(x)| = \Omega + (1-2\epsilon)_n(1-2\epsilon)g(f(x))h(x)$. Then, for $n \geq N, R_n \subset R$. In fact, if $y \in R_n, |h(y)| \leq \frac{\Omega}{(1-2\epsilon)_n}$. Therefore,

$$|f(y) + (1-2\epsilon)_n h(y)| = \Omega + (1-2\epsilon)_n(1-2\epsilon)g(f(y))h(y). \qquad (2.7)$$

So,

$\Omega + (1-2\epsilon)_n(1-2\epsilon)g(f(y))h(y) = |f(y) + (1-2\epsilon)_n h(y)| > |f(x) + (1-2\epsilon)_n h(x)| = \Omega + (1-2\epsilon)_n(1-2\epsilon)g(f(x))h(x)$and consequenctly $(1-2\epsilon)g(f(y))h(y) > (1-2\epsilon)g(f(x))h(x)$.

For all $n \geq N, R - R_n \subset \left\{y: |h(y)| > \frac{\Omega}{(1-2\epsilon)_n}\right\}$.

On the contrary, let $y \in R - R_n$ be with $|h(y)| \leq \frac{\Omega}{(1-2\epsilon)_n}$. From (2.7) we have

$|f(y) + (1-2\epsilon)_n h(y)| = \Omega + (1-2\epsilon)_n(1-2\epsilon)g(f(y))h(y) > \Omega + (1-2\epsilon)_n(1-2\epsilon)g(f(x))h(x) = |f(x) + (1-2\epsilon)_n h(x)|$, which is a contradication. Since $\mu(R - R_n) \leq \mu_h\left(\frac{\Omega}{(1-2\epsilon)_n}\right)$ we have

$$\mu(R_n) = \mu(R). \qquad (2.8)$$

Lemma 2.10

Let $f, h \in \Lambda_{\omega,\emptyset}$. If $\Omega > 0$ and $x \in C_f(\Omega)$, then

$$\mu(\{y \in C_f(\Omega): |f(y) + (1-2\epsilon)h(y)| = |f(x) + (1-2\epsilon)h(x)| \; and \; y \leq x\}) = \mu(\{y \in C_f(\Omega): (1-2\epsilon)g(f(y))h(y) = (1-2\epsilon)g(f(x))h(x) \; and \; y \leq x\}).$$

Proof. Let $(1-2\epsilon)_n \downarrow 0$,

$$R_n \coloneqq \left\{y \in C_f(\Omega): |f(y) + (1-2\epsilon)_n h(y)| = |f(x) + (1-2\epsilon)_n h(x)|, |h(y)| \leq \frac{\Omega}{(1-2\epsilon)_n} \; and \; y \leq x\right\},$$

$$(1-2\epsilon)_n \coloneqq \left\{y \in C_f(\Omega): |f(y) + (1-2\epsilon)_n h(y)| = |f(x) + (1-2\epsilon)_n h(x)|, |h(y)| > \frac{\Omega}{(1-2\epsilon)_n} \; and \; y \leq x\right\}$$

and $R \coloneqq \{y \in C_f(\Omega): (1-2\epsilon)g(f(y)h(y) = (1-2\epsilon)g(f(x)h(x) \; and \; y \leq x\}$.

Now, the proof follows in the same way as in Lemma 2.9

Theorem 2.11

Let $f, h \in \Lambda_{\omega,\emptyset}$ and let $\tau_{f,h}$ be defined by (1.2).

(a) If $x \in (f)$, $\sigma_{f+h(1-2\epsilon)}(x) = \tau_{f,h}(x)$;

(b)If $x \in (h) - (f)$, $\sigma_{f+h(1-2\epsilon)}(x) = \mu_f(0) + \mu_{h_{(h)-(f)}}(x)$.

Proof. (a) Let $x \in (f)$ and define $\Omega = |f(x)|$. We observe that for all sufficiently small $1-2\epsilon, x \in \left(f + h(1-2\epsilon)\right)$. Hence, we get

$\sigma_{(f+h(1-2\epsilon))}(x) = \mu_{(f+h(1-2\epsilon))\chi_{\underline{C_f(\Omega)}}}(|f(x) + (1-2\epsilon)h(x)|) + \mu\left(\left\{y \in \underline{C_f(\Omega)}: |f(y) + (1-2\epsilon)h(y)| = |f(x) + (1-2\epsilon)h(x)| and\ y \le x\right\}\right) + \mu\left(\left\{y \in C_f(\Omega): |f(y) + (1-2\epsilon)h(y)| > |f(x) + (1-2\epsilon)h(x)|\right\}\right) + \mu\left(\left\{y \in C_f(\Omega): |f(y) + (1-2\epsilon)h(y)| = |f(x) + (1-2\epsilon)h(x)|\ and\ y \le x\right\}\right)$.

Therefore, Lemmas 2.7 - 2.10 imply (a).

(b) Let $x \in (h) - (f)$. Suppose $\mu(f) < \infty$.

For all $\epsilon > \frac{1}{2}$, $x \in (f + (1-2\epsilon)h)$ .Then ,we have

$$\begin{aligned}\sigma_{f+(1-2\epsilon)h}(x) &= \mu(\{y \in (f) : |f(y) + (1-2\epsilon)h(y)| > (1-2\epsilon)|h(x)|\}) \\ &+ \mu(\{y \in (f) : |f(y) + (1-2\epsilon)h(y)| = (1-2\epsilon)|h(x)| and\ y \le x\}) \\ &+ \sigma_{h_{(h)-(f)}}(x). \qquad (2.9)\end{aligned}$$

According to Lemma 2.8 the second term of (2.9) tends to zero. We only need to prove that the first term tends to $\mu_f(0)$.

$$R_{(1-2\epsilon)} := \{y \in (f) : |f(y) + (1-2\epsilon)h(y)| > (1-2\epsilon)|h(x)|\},$$

$T_{(1-2\epsilon)} := \{y \in (f) : |f(y)| \le (1-2\epsilon)(|h(y)| + |h(x)|)\}$.

Since$\mu(T_{1-2\epsilon}) = 0$ and

$$(f) - R_{1-2\epsilon} \subset T_{1-2\epsilon}, \qquad (2.10)$$

then $= \mu\left((f)\right) = \mu_f(0)$.

Now, suppose that $\mu\left((f)\right) = \infty$ .Given $M > 0$, we can choose $(1-2\epsilon)_1$ such that $\mu_f(|(1-2\epsilon)_1 h(x)|) > M$ and $\mu\left(C_f(|(1-2\epsilon)_1 h(x)|)\right) = 0$.

Then by Lemma 2.5 we obtain

$= \mu_f(|(1-2\epsilon)_1 h(x)|)M$ .Thus,$\mu_{f+(1-2\epsilon)h}(|(1-2\epsilon)_1 h(x)|) > M$ for all sufficiently small $(1-2\epsilon)$. It follows that $M < \mu_{f+h(1-2\epsilon)}(|(1-2\epsilon)h(x)|)$, for all sufficiently small $(1-2\epsilon)$.

Finally, as $\mu_{f+h(1-2\epsilon)}(|(1-2\epsilon)h(x)|) \le \sigma_{f+h(1-2\epsilon)}(x)$, the proof of (b) is complete.

Definition 2.12

Let $f, h \in \Lambda_{\omega,\emptyset}$.We define

$$\rho_{f,h}(x) = \{\tau_{f,h(x) if\ x \in (f)}, \mu_f(0) + \sigma_{h_{(h)-(f)}}(x)\ if\ x \in (h) - (f) .$$

Example 2.13

Let $f, h \in \Lambda_{\omega,\emptyset}$.If $((f)) < \infty$, then $\rho_{f,h}$ is m.p.t. from $(f) \cup (h)$ onto$[0, \mu(f) \cup (h)))$ such that $|f| = f^* \circ \rho_{f,h}\mu$-a.e. on $(f) \cup (h)$ .Let $f, h \in \Lambda_{\omega,\emptyset}$ and let $(1-2\epsilon)$ be non- zero real number. We denote $F(1-2\epsilon) = \frac{\emptyset(|f+h(1-2\epsilon)|-\emptyset(|f|))}{1-2\epsilon}$.The next result is one of the main theorems of this section.

Theorem 2.14

Let $, h \in \Lambda_{\omega,\emptyset}$ .Then $\gamma^+_{\varphi_{\omega,\emptyset}}(f,h) =$

$$\int_{(f)}^{\infty} \varphi\, \omega(\rho_{f,h})\emptyset'(|f|)(1-2\epsilon)g(f)h\, d\mu + \emptyset'_+(0)\int_{(h)-(f)}^{\infty} \varphi\, \omega(\rho_{f,h})|h|\, d\mu , \qquad (2.11)$$

where $\emptyset'_+(0)$ is the right derivative of $\emptyset$ at 0. In (2.11) , we write $\omega(\infty) = 0$.

Proof. Assume that $\mu((f)) < \infty$ .Let $\epsilon > \frac{1}{2}$. Clearly

$$\varphi_{\omega,\emptyset}(f + h(1-2\epsilon)) = \int_{(f+h(1-2\epsilon))} .\omega(\sigma_{f+h(1-2\epsilon)})\emptyset(|f + h(1-2\epsilon)|)\ d\mu$$

and by Example 2.13 we have

$$\varphi_{\omega,\emptyset}(f) = \int_{(f)\ \cup (h)}^{\infty} \varphi\, \omega(\rho_{f,h})\emptyset(|f|)\, d\mu$$

As $\left(\omega(\rho_{f,h})\right)^{*} = \omega$ in $[\,0, \mu((f)\ \cup\ (h)\,))$, by the Hardy-Littlewood inequality (see [1] Bennet and Sharpley, 1988)

$$\int_{(f)\ \cup (h)}^{\infty} \varphi\, \omega(\rho_{f,h})\emptyset(|f + h(1-2\epsilon)|)\, d\mu \le \int_{0}^{\alpha} \varphi\, \omega(1+2\epsilon)\emptyset\left(\left(f + h(1-2\epsilon)\right)^{*}(1+2\epsilon)\right) d(1+2\epsilon).$$

In consequence , we get

$$\frac{1}{1-2\epsilon}\left(\int_{(f+h(1-2\epsilon))}^{\infty} \varphi\, \omega(\sigma_{f+h(1-2\epsilon)})\emptyset(|f + h(1-2\epsilon)|)\, d\mu - \int_{(f)\ \cup (h)}^{\infty} \varphi\, \omega(\rho_{f,h})\emptyset(|f + h(1-2\epsilon)|)\, d\mu\right) \ge 0.$$

Therefore

$$\frac{\varphi_{\omega,\emptyset}(f + h(1-2\epsilon)) - \varphi_{\omega,\emptyset}(f)}{1-2\epsilon} \ge \int_{(f)}^{\infty} \varphi\, \omega(\rho_{f,h})F(1-2\epsilon)\, d\mu + P(1-2\epsilon), \qquad (2.12)$$

where $P(1-2\epsilon) = \int_{(h)\,-\,(f)}^{\infty} \varphi\, \omega(\rho_{f,h})\frac{\emptyset|h(1-2\epsilon)|}{1-2\epsilon}\, d\mu.$
Analogously with$(\,f + h(1-2\epsilon))$ instead of f,we get the inequality

$$\frac{\varphi_{\omega,\emptyset}(f + h(1-2\epsilon)) - \varphi_{\omega,\emptyset}(f)}{1-2\epsilon} \le \int_{(f+h(1-2\epsilon))\cap(f)}^{\infty} \varphi\, \omega(\sigma_{f+h(1-2\epsilon)})\varphi.(1-2.\in)\, d\mu. + \rho.(1-2\in.), \qquad (2.13)$$

where $Q(1-2\epsilon) = \int_{(h)\,-\,(f)}^{\infty} \varphi\, \emptyset(.\varphi_{f+h(1-2\epsilon)})\frac{\emptyset|h(1-2\epsilon)|}{1-2\epsilon}\, d\mu.$
Let $\frac{1}{2} < \epsilon\ < 0$. Since $\emptyset$ is a convex function, we have $\emptyset'(1+2\epsilon)1+2\epsilon \le \emptyset(2(1+2\epsilon))$ for all $\epsilon \ge \frac{-1}{2}$.
In addition, the mean value Theorem implies that
$|F(1-2\epsilon)| \le \emptyset'(\{|f + h(1-2\epsilon)|, |f|\}\,)|h|.$
Therefore $\left|\omega\left((\rho_{f,h})\right)F(1-2\epsilon)\right| \le \omega(\rho_{f,h})\emptyset(2(|f| + |h|)).$
Also, for all sufficiently small $(1-2\epsilon), \epsilon > \frac{1}{2}$, from the proof of Theorem 2.11 we obtain $\rho_{f,h} \le \sigma_{(f+h(1-2\epsilon))}$.So, $\left|\omega(\sigma_{f+h(1-2\epsilon)})\, F(1-2\epsilon)\right| \le \omega(\rho_{f,h})\emptyset(2(|f| + |h|))$ for all sufficiently small $(1-2\epsilon)\,, \epsilon > \frac{1}{2}$. Clearly,

$$\int_{(f)}^{\infty} \varphi\, \omega(\rho_{f,h})\emptyset(2(|f| + |h|))d\mu \le \int_{0}^{\alpha} \varphi\, \omega\emptyset(2(|f| + |h|)^{*})d(1+2\epsilon) < \infty,$$

then, from Theorem 2.11 and the Lebesgue Convergence Theorem we get

$$\int_{(f)}^{\infty} \varphi\, \omega(\rho_{f,h})F(1-2\epsilon)\, d\mu = \int_{(f)}^{\infty} \varphi\, \omega(\rho_{f,h})\emptyset'\,(|f|)(1-2\epsilon)g(f)h\, d\mu \qquad (2.14)$$

and

$$\int_{(f+h(1-2\epsilon))\cap(f)}^{\infty} \varphi\, \omega(\sigma_{f+h(1-2\epsilon)})F(1-2\epsilon)\, d\mu = \int_{(f)}^{\infty} \varphi\, \omega(\rho_{f,h})\emptyset'\,(|f|)(1-2\epsilon)g(f)h\, d\mu. \qquad (2.15)$$

If $((h) - (f) = 0$, then $P(1-2\epsilon) \equiv 0$ and $Q(1-2\epsilon) \equiv 0$. So,(2.11) holds. Otherwise, for all sufficiently small $(1-2\epsilon)$, $\epsilon > \frac{1}{2}$,

$$\omega(\rho_{f,h})\frac{\emptyset(|h(1-2\epsilon)|)}{1-2\epsilon} \leq \omega\left(\sigma_{h_{(h)-(f)}}\right)\emptyset(|h|) on (h) - (f).$$

Since

$$\int_{(h)-(f)}^{\infty} \varphi\, \omega\left(\sigma_{h_{(h)-(f)}}\right)\emptyset(|h|) \leq \varphi_{\omega,\emptyset}(h) < \infty, \qquad (2.16)$$

the Lebesgue Convergence Theorem implies that

$$P(1-2\epsilon) = \emptyset'_{+}(0)\int_{(h)-(f)} \varphi\, \omega(\rho_{f,h})|h| d\mu. \qquad (2.17)$$

On the other hand ,

$$\omega(\sigma_{f+h(1-2\epsilon)})\frac{\emptyset(|h(1-2\epsilon)|)}{1-2\epsilon} = \emptyset'_{+}(0)\, \omega(\rho_{f,h})|h| on (h) - (f)$$

and for all sufficiently small $(1-2\epsilon)$, $\epsilon > \frac{1}{2}$, (2.9) implies

$$\omega(\sigma_{f+h(1-2\epsilon)})\frac{\emptyset(|h(1-2\epsilon)|)}{1-2\epsilon} \leq \omega\left(\sigma_{h_{(h)-(f)}}\right)\emptyset(|h|) on\ (h) - (f). \quad (2.18)$$

According to (2.16) and the Lebesgue Convergence Theorem ,

$$Q(1-2\epsilon) = \emptyset'_{+}(0)\int_{(h)-(f)} \varphi\, \omega(\rho_{f,h})|h| d\mu. \qquad (2.19)$$

Therefore,(2.14) ,(2.15) ,(2.17) and(2.19) imply (2.11) .
Now assume that $\mu((f)) = \infty$ .Similarly, to the proof of (2.12) and(2.13) ,we can obtain $\int_{(f)} \omega(\rho_{f,h}) F(1-2\epsilon) d\mu \leq \frac{\varphi_{\omega,\emptyset}(f+h(1-2\epsilon)) - \varphi_{\omega,\emptyset}(f)}{1-2\epsilon} \leq$

$$\int_{(f+h(1-2\epsilon))\cap(f)} \varphi\, \omega(\rho_{f,h}) F(1-2\epsilon) d\mu + Q(1-2\epsilon). \qquad (2.20)$$

Proceeding as before , we get (2.14) and (2.15). If $\mu((h) - (f) = 0$, $Q(1-2\epsilon) \equiv 0$ and (2.11) is true. In opposite case ,$w(\sigma_{f+h(1-2\epsilon)}) \frac{\emptyset(|h(1-2\epsilon)|)}{1-2\epsilon} = 0$.
From (2.16),(2.18) and the Lebesgue Convergence Theorem,$Q(1-2\epsilon) = 0$.The proof is complete. In the next theorem, we obtain the one-sided Gateaux derivative of the Lebesgue norm in terms of the one-sided Gateaux derivative of the functional $\varphi_{\omega,\emptyset}$.

Theorem 2.15

Let $f, h \in \Lambda_{\omega,\emptyset}$, $f \neq 0$ .Then

$$\gamma^{+}_{\|\cdot\|_{\omega,\emptyset}}(f,h) = \frac{\gamma^{+}_{\varphi_{\omega,\emptyset}}\left(\frac{f}{\|f\|_{\omega,\emptyset}}, h\right)}{\gamma^{+}_{\varphi_{\omega,\emptyset}}\left(\frac{f}{\|f\|_{\omega,\emptyset}}, \frac{f}{\|f\|_{\omega,\emptyset}}\right)}. \qquad (2.21)$$

Proof. If $h = 0$, (2.21) is obvious. Now suppose that $h \neq 0$.
For all $(1-2\epsilon)$, $0 < 1-2\epsilon < \frac{\|f\|_{\omega,\emptyset}}{2\|h\|_{\omega,\emptyset}}$, we denote
$K(1-2\epsilon) = \frac{\emptyset\left(\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}\right) - \emptyset\left(\frac{|f|}{\|f\|_{\omega,\emptyset}}\right)}{1-2\epsilon}$ and $G(1-2\epsilon) = \varphi_{\omega,\emptyset}\left(\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}\right)$.
First , we assume that $\mu((f)) < \infty$ and we consider

$P(1-2\epsilon)=\int_{(h)\,-(f)}\varphi\,\omega(\rho_{f,h})\frac{\emptyset\left(\frac{|h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}\right)}{1-2\epsilon}d\mu$ and

$$Q(1-2\epsilon)=\int_{(h)\,-(f)}\varphi\,\omega(\sigma_{f+h(1-2\epsilon)})\frac{\emptyset\left(\frac{|h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}\right)}{1-2\epsilon}d\mu.$$

Proceeding analogously to the proof of Theorem 2.14,we can obtain

$$\int_{(f)}\varphi\,\omega(\rho_{f,h})K(1-2\epsilon)d\mu+P(1-2\epsilon)\le\frac{G(1-2\epsilon)-G(0)}{1-2\epsilon}\le$$

$$\int_{(f+h(1-2\epsilon))\cap(f)}\varphi\,\omega(\sigma_{f+h(1-2\epsilon)})K(1-2\epsilon)d\mu+Q(1-2\epsilon), \qquad (2.22)$$

Let $0<1-2\epsilon\le\left\{1,\frac{\|f\|_{\omega,\emptyset}}{2\|h\|_{\omega,\emptyset}}\right\}$.

Adding and subtracting $\frac{|f|\|f\|_{\omega,\emptyset}}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}\|f\|_{\omega,\emptyset}}$ to the expression $\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}-\frac{|f|}{\|f\|_{\omega,\emptyset}}$ and applying the triangular inequality we obtain

$$\left|\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}-\frac{|f|}{\|f\|_{\omega,\emptyset}}\right|\le\frac{2(1-2\epsilon)M}{\|f\|_{\omega,\emptyset}}(|f|+|h|),$$

where $M=1+\frac{\|h\|_{\omega,\emptyset}}{\|f\|_{\omega,\emptyset}}$.In consequence, the Main Value Theorem implies that

$$|K(1-2\epsilon)|\le\frac{\emptyset'\left(\left\{\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}},\frac{|f|}{\|f\|_{\omega,\emptyset}}\right\}\right)}{1-2\epsilon}\left|\frac{|f+h(1-2\epsilon)|}{\|f+h(1-2\epsilon)\|_{\omega,\emptyset}}-\frac{|f|}{\|f\|_{\omega,\emptyset}}\right|$$
$$\le\emptyset'\left(\frac{2M}{\|f\|_{\omega,\emptyset}}(|f|+|h|)\right)\frac{2M}{\|f\|_{\omega,\emptyset}}(|f|+|h|)\le\emptyset\left(\frac{4M}{\|f\|_{\omega,\emptyset}}(|f|+|h|)\right).$$

From the Lebesgue Convergence Theorem, we can show that

$$\int_{(f)}\varphi\,\omega(\rho_{f,h})K(1-2\epsilon)d\mu=\int_{(f+h(1-2\epsilon))\,\cap(f)}\varphi\,\omega(\sigma_{f+h(1-2\epsilon)})K(1-2\epsilon)d\mu$$
$$=\int_{(f)}\varphi\,\omega(\rho_{f,h})\emptyset'\left(\frac{|f|}{\|f\|_{\omega,\emptyset}}\right)\left(\frac{(1-2\epsilon)g(f)h}{\|f\|_{\omega,\emptyset}}-\frac{|f|}{\|f\|^2_{\omega,\emptyset}}\gamma^+_{\|\cdot\|_{\omega,\emptyset}}(f,h)\right)d\mu \qquad (2.23)$$

and

$$\lim_{1-2\epsilon\to0^+}P(1-2\epsilon)=\lim_{1-2\epsilon\to0^+}Q(1-2\epsilon)=\emptyset'_+(0)\int_{(h)\,-(f)}\varphi\,\omega(\rho_{f,h})\frac{|h|}{\|f\|_{\omega,\emptyset}}d\mu. \qquad (2.24)$$

Thus, from (2.22)-(2.24) , we have

$$\lim_{1-2\epsilon\to 0^+}\frac{G(1-2\epsilon)-G(0)}{1-2\epsilon}$$
$$=\emptyset'_+(0)\int_{(h)\,-(f)}\varphi\,\omega(\rho_{f,h})\frac{|h|}{\|f\|_{\omega,\emptyset}}d\mu$$
$$+\int_{(f)}\varphi\,\omega(\rho_{f,h})\emptyset'\left(\frac{|f|}{\|f\|_{\omega,\emptyset}}\right)\left(\frac{(1-2\epsilon)g(f)h}{\|f\|_{\omega,\emptyset}}-\frac{|f|}{\|f\|^2_{\omega,\emptyset}}\gamma^+_{\|\cdot\|_{\omega,\emptyset}}(f,h)\right)d\mu. \quad (2.25)$$

Since $\varphi_{\omega,\emptyset}\left(\frac{g}{\|g\|_{\omega,\emptyset}}\right)=1$ for any $g\in \Lambda_{\omega,\emptyset}-\{0\}$(see [6] Kamińska,1990) we have

$$G(1-2\epsilon)=1$$

for $0\le 1-2\epsilon<\frac{\|f\|_{\omega,\emptyset}}{2\|h\|_{\omega,\emptyset}}$.Therefore, from (2.25) and Theorem 2.14, we get (2.21) .

The case $\mu((f))=\infty$ follows in a similar way without using $P(1-2\epsilon)$ and after proving that $Q(1-2\epsilon)=0$.

3. characterization of smooth points for the Luxemburg norm

We let X be a Banach space and let $T: X\to R^+$ be a convex functional.The following example shows that the set of smooth points of the functional T, in general, is not equal to the set of smooth points of the Minkowski functional of $\{f\in X: T(f)\le 1\}$)[12] Levis and Cuenya, 2007).

Example 3.1

In a Hilbert space X define the continuous convex function

$$T(f)=\{\|f\| \quad if \quad \|f\|>1, 1 \quad if \quad \|f\|\le 1.$$

It is not Gateaux differentiable at any point f of norm |, but the Minkowski functional of $\{f\in X: T(f)\le 1\}$. (which is the closed unit ball) is just the norm , which is infinitely Gateaux differentiable everywhere except at the origin.We consider the sets

$$E^{\omega,\emptyset}:=\left\{f\in \Lambda_{\omega,\emptyset}-\{0\}: \mu\{|f|=1-2\epsilon\}=0\ for\ any\ \epsilon>\tfrac{1}{2}\right\}\text{ and}$$
$$\Delta^{\omega,\emptyset}:=E^{\omega,\emptyset}\cap\left\{f\in \Lambda_{\omega,\emptyset}: \mu\{f=0\}=0\ \ on\ \mu_f(0)=\infty\right\}.$$

In ([10] Levis and Cuenya, 2004),we have proved that $f\in \Lambda_{\omega,\emptyset}$ is a smooth points of $\varphi_{\omega,\emptyset}$ if $f\in E^{\omega,\emptyset}(f\in \Delta^{\omega,\emptyset})$ when $\emptyset'_+(0)=0(\emptyset'_+(0)>0)$.

It is well-known that if X is a Banach space and $T: X\to R$ is a convex functional then for all $f,h\in X$, $\gamma^+_T(f,h)$and$\gamma^-_T(f,h)$always exist and the equality $\gamma^+_T(f,h)=-\gamma^-_T(f,-h)$holds (see [14] Pinkus,1989),we showed arelation between the one-sided Gateaux derivative for the functional $\varphi_{\omega,\emptyset}$ and the one-sided Gateaux derivative for the Luxemburg norm .Consequently, f is a smooth point of the Luxemburg norm if and only if $\frac{f}{\|f\|_{\omega,\emptyset}}$ is a smooth point for $\varphi_{\omega,\emptyset}$ .The next theorm follows immediately.

Theorem 3.2

The set of smooth points for the Luxemburg norm is $E^{\omega,\emptyset}(\Delta^{\omega,\emptyset})$ if $\emptyset'_+(0)=0(\emptyset'_+(0)>0)$.

Remark 3.3

It is well-known that $E^{\omega,\emptyset}$ and $\Delta^{\omega,\emptyset}$ are dense sets in the $\Lambda_{\omega,\emptyset}$ because the points of Gateaux- differentiability of the norm in a separable space always from a dense set (see[13] Phelps,1989).

4.Characterization of best approximants

We characterize the set of best approximants from convex closed sets using the one-sided Gateaux derivative. Moreover, we establish a relation between the best $\varphi_{\omega,\emptyset}$-approximants and best approximants from a convex set.Let $f, h \in \Lambda_{\omega,\emptyset}$.We denote by

$$A_f := \{\sigma : (f) \to (f^*) : \sigma \, is \, m.p.t \, . \, and \, |f| = f^* \circ \sigma \, , \mu - a.e. on(f)\}$$

and

$$T_{f,h} = -\emptyset'_+(0) \int_{(h)-(f)} \varphi \, \omega(\rho_{f,h})|h| d\mu. \qquad (4.1)$$

In (4.1), we write $\omega(\infty) = 0$.

Theorem 4.1

Let $K \subset \Lambda_{\omega,\emptyset}$ be a convex closed set, let $f, h \in \Lambda_{\omega,\emptyset} - K$ and let $h^\bullet \in K$. Then the following statements are equivalent :

(a) $h^\bullet \in P_{\varphi_{\omega,\emptyset}}(f, K)$;

(b) $\int_{(f-h^\bullet)}^{\alpha} \varphi \, \omega(\rho_{f-h^\bullet,h^\bullet-h})\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu \geq T_{f-h^\bullet,h^\bullet-h} for \, all \, h \in K;$

(c) $\int_{(f-h^\bullet)}^{\alpha} \varphi \, \omega(\sigma)\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu \geq T_{f-h^\bullet,h^\bullet-h} \, for \, all \, h \in K.$

In addition, if $T_{f-h^\bullet,h^\bullet-h=0}$,these statements imply

(d)$\varphi_{\omega,\emptyset}(f-h^\bullet) \leq \int_0^{\alpha} \varphi \, \omega(1+2\epsilon)\emptyset'((f-h^\bullet)^*(1+2\epsilon))(f-h)^*(1+2\epsilon)d(1+2\epsilon)$ for all $h \in K$.

Proof. The implication(b) ⇒ (c) is obvious.

(a)⇔(b). This is an immediate consequence of Theorem 2.14 and ([14] Pinkus,1989,Theorem 1.6), because this theorem still holds if we replace the norm$\|\cdot\|$ by the functional $\varphi_{\omega,\emptyset}$.

(c)⇒(b). Let $h \in K$ and $\sigma \in A_{f-h^\bullet}$ .Taking σ , $f - h^\bullet$ and $h^\bullet - h$ instead of $\rho_{f,h}, f$ and h respectively in (2.12) or (2.20) we have

$$\int_{(f-h^\bullet)}^{\alpha} \varphi \, \omega(\sigma)\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu - T_{f-h^\bullet,h^\bullet-h} \leq$$

$$\Upsilon^+_{\varphi_{\omega,\emptyset}}(f-h^\bullet)(h^\bullet-h).$$

By hypothesis and Theorem 2.14, we get

$$0 \leq \int_{(f-h^\bullet)} \varphi \, \omega \, (\sigma)\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu - T_{f-h^\bullet,h^\bullet-h} \leq \Upsilon^+_{\varphi_{\omega,\emptyset}}(f-h^\bullet, h^\bullet-h)$$

$$= \int_{(f-h^\bullet)} \varphi \, \omega(\rho_{f-h^\bullet,h^\bullet-h})\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(h^\bullet-h)d\mu - T_{f-h^\bullet,h^\bullet-h}.$$

(b)⇒(d) .Assume $T_{f-h^\bullet,h^\bullet-h} = 0$ and let $h \in K$ .Since for all $\epsilon > \frac{-1}{2}$, $\emptyset(1+2\epsilon) \leq \emptyset'(1+2\epsilon)1+2\epsilon$, then by hypothesis and the Hardy-Littlewood inequality we have

$$\varphi_{\omega,\emptyset}(f-h^\bullet) \leq \int_{(f-h^\bullet)} \varphi \, \omega(\rho_{f-h^\bullet,h^\bullet-h})\emptyset'(|f-h^\bullet|)|f-h^\bullet|d\mu$$

$$\leq \int_{(f-h^\bullet)} \varphi \, \omega(\rho_{f-h^\bullet,h^\bullet-h})\emptyset'(|f-h^\bullet|)(1-2\epsilon)g(f-h^\bullet)(f-h)d\mu.$$

According to ([1] Bennet and Sharpley,1988,Proposition7.2) $\omega(\rho_{f-h^\bullet,h^\bullet-h})\emptyset'(|f-h^\bullet|) \sim \omega\emptyset'((f-h^\bullet)^*)$ and $\omega\emptyset'((f-h^\bullet)^*)$ is anon-increasing function. So,

$$\varphi_{\omega,\emptyset}(f-h^\bullet) \leq \int_0^{\propto} \varphi \, \omega\emptyset'((f-h^\bullet)^*(1+2\epsilon))(f-h)^*(1+2\epsilon)d(1+2\epsilon).$$

The following example shows , that the implication (d)⇒(a) of Theorem 4.1 is not true in general.

Example 4.2

Let $\alpha = 1$. We consider $\phi(1+2\epsilon) = \{e^{(1+2\epsilon)} - 2(1+\epsilon)\ if\ \frac{-1}{2} \le \epsilon \le 0, (e-2)(1+2\epsilon)^{\frac{e-1}{e-2}}\ if\ \epsilon > 0$.

$K \coloneqq \{h \in \Lambda_{\omega,\phi}: h\ is\ constant\}$and $f = \chi_{[0,\frac{1}{2})}$.It is easy to see that $\varphi_{\omega,\phi}(f) = (e-2)\int_0^{\frac{1}{2}}.\omega(1+2\epsilon)d(1+2\epsilon)$.On the other hand, for $h \in K$ and $\frac{-1}{2} \ge \epsilon > \frac{-1}{4}$, we have$(f-h)^*(1+2\epsilon) \ge \frac{1}{2}$ .Thus, for all $h \in K$, $\int_0^1 \quad \omega(1+2\epsilon)\phi'(f^*(1+2\epsilon))(f-h)^*(1+2\epsilon)d(1+2\epsilon) \ge \frac{e-1}{2}\int_0^{\frac{1}{2}}.\omega(1+2\epsilon)d(1+2\epsilon)$.Consequently,(d) is true for $h^\bullet = 0$. However, $P_{\varphi_{\omega,\phi}}(f,K) = \left\{\frac{1}{2}\right\}$. Nevertheless, we show in the next theorem that (d)⇒(a) holds when $\phi(1+2\epsilon) = (1+2\epsilon)^{1+\epsilon}$.

Theorem 4.3

Let $K \subset L_{(\omega,1+\epsilon)}$ be a convex closed set , let $0 \le \epsilon < \infty$,
let $f \in L_{(\omega,1+\epsilon)} - K$ and let $h^\bullet \in K$.Then, the following statements are equivalent :

(a) $h^\bullet \in P_{\|\cdot\|_{(\omega,1+\epsilon)}}(f,k)$.

(b) $\|f-h^\bullet\|_{(\omega,1+\epsilon)}^{(1+\epsilon)} \le \int_0^\alpha.\omega(1+2\epsilon)\left((f-h^\bullet)^*(1+2\epsilon)\right)^\epsilon(f-h)^*(1+2\epsilon)d(1+2\epsilon)\ for\ all\ h \in K.$

Proof. If $\epsilon = 0$, this is obvious. Assume that $0 < \epsilon < \infty$. (a)⇒(b) is an immediate consequence of Theorem 4.1.

(b)→(a) .Let $(1+\epsilon)$ and $(1-\epsilon)$ be conjugate numbers and let $h \in K$.From hypothesis and Hölder inequality, we get

$$\begin{aligned}\|f-h^\bullet\|_{(\omega,1+\epsilon)}^{(1+\epsilon)} &\le \int_0^\alpha.\omega(1+2\epsilon)\left((f-h^\bullet)^*(1+2\epsilon)\right)^\epsilon(f-h)^*(1+2\epsilon)d(1+2\epsilon)\\ &= \int_0^\alpha.\omega(1+2\epsilon)^{\frac{1}{1-\epsilon}}\left((f-h^\bullet)^*(1+2\epsilon)\right)^\epsilon\omega(1+2\epsilon)^{\frac{1}{1+\epsilon}}(f-h)^*(1+2\epsilon)d(1+2\epsilon)\\ &\le \|f-h^\bullet\|_{(\omega,1+\epsilon)}^\epsilon\|f-h\|_{(\omega,1+\epsilon).}\end{aligned}$$

So, the proof is complete.Next, we establish a relation between the best $\varphi_{\omega,\emptyset}$-approximants and best approximants from a convex set K.

Theorem 4.4

Let $f,\in \Lambda_{\omega,\emptyset}$ and $K \subset \Lambda_{\omega,\emptyset}$ be convex set such that
$\delta = E_{\|\cdot\|_{\omega,\emptyset}}(f,K) > 0$.Then, $h^\bullet \in P_{\|\cdot\|_{\omega,\emptyset}}(f,K)$if and only if $\frac{h^\bullet}{\delta} \in P_{\varphi_{\omega,\emptyset}}\left(\frac{f}{\delta},\frac{K}{\delta}\right)$.

Proof. It follows immediately from ([14] Pinkus,1989, Theorem 1.6 and Theorem 2.15).

Remark 4.5

Theorem 4.4 is known for arbitrary sets K in modular space (see [9] Kilmer and Kozlowski,1990).
Henceforth, we consider $\alpha < \infty, K \coloneqq \{g \in \Lambda_{\omega,\emptyset}: g\ is\ constant\}$ and
$f \in \Lambda_{\omega,\emptyset}$.Clearly,$P_{\varphi_{\omega,\emptyset}}(f,k)$is anonempty and compact interval.We denote
$\underline{f} = P_{\varphi_{\omega,\emptyset}}(f,k)$ and $\underline{f} = P_{\varphi_{\omega,\emptyset}}(f,k)$.
As a direct consequence of ([14] Pinkus,1989, Theorem 1.6) we have that $c \in P_{\varphi_{\omega,\emptyset}}(f,k)$ if and only if

$$\gamma_{\varphi_{\omega,\emptyset}}^+(f-c,1) \ge 0\ and\ \gamma_{\varphi_{\omega,\emptyset}}^+(c-f,1) \ge 0. \quad (4.2)$$

The next characterization of best constant $\varphi_{\omega,\emptyset}$-approximants of f follows from (4.2) and Theorem 2.14.

Theorem 4.6

Let $f,\in \Lambda_{\omega,\emptyset}$. Then $c \in P_{\varphi_{\omega,\emptyset}}(f,k)$ if and only if the following statements hold:

(a)$\int_{f\ge c}.\omega(\rho_{f-c,1})\emptyset'(f-c)d\mu \ge \int_{f<c}.\omega(\rho_{f-c,1})\emptyset'(c-f)d\mu$

and

(b)$\int_{f\le c}.\omega(\rho_{c-f,1})\emptyset'(c-f)d\mu \ge \int_{f>c}.\omega(\rho_{c-f,1})\emptyset'(f-c)d\mu$.

We write, $\emptyset'(0) \coloneqq \emptyset'_{+}(0)$.According to Theorem 4.4 and 4.6,we obtain the following characterization of best constant approximants :

Corollary 4.7

Let $f \in \Lambda_{(\omega,1+\epsilon)} - K$.Then $c \in P_{\|\cdot\|_{\omega,\emptyset}}(f,k)$if and only if the following statements hold:

(a) $\int_{f\geq c} .\,\omega(\rho_{f-c,1})\emptyset'\left(\frac{f-c}{\|f-c\|_{\omega,\emptyset}}\right)d\mu \geq \int_{f<c} .\,\omega(\rho_{f-c,1})\emptyset'\left(\frac{c-f}{\|c-f\|_{\omega,\emptyset}}\right)d\mu$ and

(b) $\int_{f\leq c} .\,\omega(\rho_{c-f,1})\emptyset'\left(\frac{c-f}{\|c-f\|_{\omega,\emptyset}}\right)d\mu \geq \int_{f>c} .\,\omega(\rho_{c-f,1})\emptyset'\left(\frac{f-c}{\|f-c\|_{\omega,\emptyset}}\right)d\mu.$

Now our purpose is to give away to construct the best $\varphi_{\omega,\emptyset}$-approximants $\underline{f}$ and $\underline{f}$.

We begin with three lemmas.

Lemma 4.8

If $c < d$ then for all $0 \leq x \leq \alpha$ we have

(a) $\mu_{f-c}(f(x) - c) \leq \mu_{f-d}(f(x) - d) if\ \ f(x) \geq d$ and

(b) $\mu_{f-d}(d - f(x)) \leq \mu_{f-c}(c - f(x)) if\ \ f(x) < c.$

Proof. (a) Suppose $f(x) \geq d$, clearly,

$$\mu(\{y: 2d - f(x) \leq f(y) \leq f(x)\}) \leq \mu(\{y: 2c - f(x) \leq f(y) \leq f(x)\}).$$

Therefore, $\mu(\{y: |f(y) - d| \leq f(x) - d\}) \leq \mu(\{y: |f(y) - c| \leq f(x) - c\})$
and consequently (a) holds.

(b).Now suppose that $(x) < c$. Clearly

$\mu(\{y: |f(y) - c| \leq c - f(x)\}) \leq \mu(\{y: |f(y) - d| \leq d - f(x)\})$ and thus (b) is true.

Lemma 4.9

If $c < d$ then for $0 \leq x \leq \alpha$ we have:

(a) $\rho_{f-c,1}(x) \leq \rho_{f-d,1}(x)\ if\ \ f(x) \geq d$ and

(b) $\rho_{f-d,1}(x) \leq \rho_{f-c,1}(x)\ if\ \ f(x) < c.$

Proof. (a).Suppose $f(x) > d$. Since $f(x) > c$, from Lemma 4.8 we get

$$\rho_{f-d,1}(x) = \mu_{f-d}(f(x) - d) + \mu(\{y: f(y) = f(x) and\ y \leq x\})$$
$$\geq \mu_{f-c}(f(x) - c) + \mu(\{y: f(y) = f(x) and\ y \leq x\}) = \rho_{f-c,1}(x).$$

Now suppose that $f(x) = d$.

As$\mu_{f-c}(d - c) \leq \mu_{f-d}(0)\ and\ x \in (f - c)\ ,then\ \rho_{f-c,1}(x) = \mu_{f-c}(d - c) + \mu(\{y: f(y) = d\ and\ y \leq x\}) \leq \mu_{f-d}(0) + \mu(\{y: f(y) = d\ and\ y \leq x\}) = \rho_{f-d,1}(x).$

(b). Assume $f(x) < c$. Since

$$\mu_{f-d}(d - f(x)) + \mu(\{y: f(y) - d = d - f(x)\}) \leq \mu_{f-c}(c - f(x)),$$

we have $\rho_{f-d,1}(x) \leq \mu_{f-c}(c - f(x)) + \mu(\{y: f(y) = f(x)\ and\ y \leq x\}) \leq \rho_{f-c,1}(x).$

Lemma 4.10 Let $f \in \Lambda_{(\omega,1+\epsilon)}$. If $c \leq d$, then

(a)$\gamma^{+}_{\varphi_{\omega,\emptyset}}(f - d, 1) \leq \gamma^{+}_{\varphi_{\omega,\emptyset}}(f - c, 1)$ and

(b) $\gamma^{+}_{\varphi_{\omega,\emptyset}}(c - f, 1) \leq \gamma^{+}_{\varphi_{\omega,\emptyset}}(d - f, 1).$

Proof. (a) We will show that $\gamma^{+}_{\varphi_{\omega,\emptyset}}(f - d, 1) \leq \gamma^{+}_{\varphi_{\omega,\emptyset}}(f - c, 1) if\ c < d$.We define

$$P(u) = \int_{f\geq u}^{\cdot} .\,\omega(\rho_{f-u,1})\emptyset'(f - u)d\mu\ \ and\ \ Q(u) = \int_{f<u}^{\cdot} .\,\omega(\rho_{f-u,1})\emptyset'(u - f)d\mu.$$

Clearly$\gamma^{+}_{\varphi_{\omega,\emptyset}}(f - u, 1) = P(u) - Q(u)$.It will be sufficient to prove that P is a non-increasing function and Q is a non-decreasing function. Since ω is non-increasing $\emptyset'_{+}$ is a non-decreasing and $\{y: f(y) \geq d\} \subset \{y: f(y) > c\}$, then from Lemma 4.9 (a) we have

$P(d) \leq \int_{f>c}^{\cdot} .\,\omega(\rho_{f-c,1})\emptyset'(f - c)d\mu \leq P(c)$. Similarly, from Lemma 4.9 (b),

$$Q(c) \leq \int_{f<d}^{\cdot} .\,\omega(\rho_{f-d,1})\emptyset'(d - f)d\mu = Q(d).$$

(b). Replacing in (a),f, c and d by $-f, -d$ and$-c$ respectively, we obtain (b).

Theorem 4.11 Let $f \in \Lambda_{(\omega,1+\epsilon)}$.Then
$\underline{f} = \{c: \gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1) \geq 0\}$ and $\underline{f} = \{c: \gamma^{+}_{\varphi_{\omega,\emptyset}}(c-f,1) \geq 0\}$.

Proof. Suppose that there exists $c, c > \underline{f}$, such that

$$\gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1) \geq 0. \qquad (4.3)$$

By Lemma 4.10,

$$\gamma^{+}_{\varphi_{\omega,\emptyset}}(c-f,1) \geq \gamma^{+}_{\varphi_{\omega,\emptyset}}\left(\underline{f}-f,1\right) \geq 0. \qquad (4.4)$$

Then,(4.2)-(4.4) imply that $c \in P_{\varphi_{\omega,\emptyset}}(f,k)$, a contradiction.Thus,
$\underline{f} = \{c: \gamma^{+}_{\varphi_{\omega,\emptyset}}(f-c,1) \geq 0\}$.
Similarly, we can see that $\underline{f} = \{c: \gamma^{+}_{\varphi_{\omega,\emptyset}}(c-f,1) \geq 0\}$)[12] Levis and Cuenya, 2007).

5. Conclusion

By Letting $\Lambda_{\omega,\emptyset}$ be the Orlicz-Lorentz space.We study Gateaux differentiabilitiy of the functional $\varphi_{\omega,\emptyset}(f) = \int_0^\infty .\emptyset(f^*)\omega$ and of the Luxemburg norm. More precisely,we obtain the one-sided Gateaux derivatives in both cases and we characterize those points where the Gateaux derivative of norm exists. We give a characterization of best $\varphi_{\omega,\emptyset}$-approximants from convex closed subsets and we establish a relation between best $\varphi_{\omega,\emptyset}$-approximants and best approximants from a convex set.A characterization of best constant φ-approximants and the algorithm to construct the best constant for maximum and minimum $\varphi_{\omega,\emptyset}$-approximants are given.

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