RESPONSE CHARACTERISTICS OF COMPLEX LAG ELEMENTS IN SYSTEMS

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Abstract
Second-order systems are commonly used in control engineering due to their simplicity, as they can be easily modeled with straightforward differential equations, and their ability to accurately represent various physical systems like mass-spring-damper setups. They facilitate stability analysis through key parameters such as damping ratio and natural frequency, which help predict system responses using techniques like root locus and Nyquist plots. Moreover, their predictable frequency responses aid in controller design and tuning, while allowing for optimization of performance metrics like rise and settling time without compromising stability. Widely understood control methods, particularly PID control, enhance the ease of designing for these systems, making them suitable for many industrial applications. Overall, second-order systems provide a practical balance between simplicity, effectiveness, and the capacity to represent real-world dynamics, serving as a foundation for tackling more complex engineering challenges.
Keywords: Ramp Input; Step Input; First Derivative Error Compensation; Damping Ratio; Damped and Undamped Frequency; Second Order Response.

1. Introduction

Differentiation, integration, and a wide array of differential equations are commonly employed in the intricate mathematical analysis of numerical solutions to complex problems across various fields, including physics, engineering (such as mechanical, electrical, and industrial mechatronics), economic analysis, biological sciences, statistical data analysis, and computer science. Examples of common applications of differential equations in advanced mathematical analysis include: (i) Thermodynamic modeling, flood frequency forecasting, age-related shocks, and acoustic wave analysis; (ii) Nonlinear dynamical systems and the resolution of partial differential equations within multidimensional contexts; (iii) The mathematical analysis of sophisticated financial strategies like factor options and other derivatives; (iv) Examination of social phenomena through mathematical discovery and diffusion models; (v) Data analysis and the extraction of statistical insights in data science.

Various complex mathematical analysis techniques are employed to solve differential equations in these contexts, utilizing numerous tools such as finite difference methods, graphing, numerical concentration definitions, and more.

Lead and lag compensators are extensively used in control systems. A lead compensator can enhance a system's stability or response speed, while a lag compensator can decrease (though not entirely eliminate) the steady-state error. Based on the desired outcomes, one or more lead and lag compensators can be employed in various combinations.

Typically, lead and lag compensators for a system are designed in the form of a transfer function [1] – [6].

2. Literature Review on the Response of Complex Lag Elements

Complex lag elements are critical components in system dynamics and control theory, encompassing a combination of lag and lead characteristics manifested in various engineering applications. The modeling and analysis of these elements play a vital role in understanding the behavior of systems influenced by delays and phase shifts. This literature review seeks to explore the theoretical foundations, mathematical representations, and empirical studies surrounding the response of complex lag elements, particularly in the context of control systems, signal processing, and mechanical systems [7], [8], [9], [10], and [11].

The concept of lag in dynamic systems has been historically tied to time delays observed in physical processes. A complex lag element typically combines a straightforward lag with an added phase shift, creating a transfer function represented in the Laplace domain. The complex nature of these lag elements results from their ability to characterize systems where feedback loops or interactions introduce both time-delay and phase alteration. Researchers such as G. Franklin et al. (2015) have emphasized that understanding these elements is pivotal for stability analysis and controller design [12], [13], [14], [15] and [16].

Mathematical modeling of complex lag elements often employs a variety of techniques, particularly in the context of transfer functions and state-space representations. One notable approach is the use of Padé approximations, which provides a rational function that approximates the time delay in a system. This approximation is crucial for facilitating control strategies as it allows engineers to translate time delays into manageable forms within the feedback loop (R. H. E. Smith, 2017). Furthermore, methods such as frequency response analysis and root locus techniques are commonly utilized to assess system stability and performance. The literature indicates that these techniques are instrumental in designing robust controllers capable of handling uncertainties prevalent in systems governed by complex lag dynamics [17] and [18].

Various empirical studies have explored the response of complex lag elements across a range of applications, from mechanical systems to electronic circuits. For instance, in the study of automotive systems, complex lag elements emerge in the control of engine dynamics, where delays in throttle response and vehicle speed must be managed for optimal performance (J. D. McGowan, 2018). Analogous research has been conducted regarding aircraft flight control systems, where the identification and compensation for time delays are crucial for maintaining stability during maneuvering (K. J. Astrom & H. K. M. Witten mark, 2019). These empirical studies highlight the real-world significance of comprehending the dynamics of complex lag elements, as variations in response time can severely impact operational efficacy and safety [19], [20] and [21].

Despite the advancements in understanding complex lag elements, several challenges remain. One prominent issue is the difficulty in accurately modeling time delays, particularly in nonlinear systems where the lag might change dynamically based on operating conditions. Furthermore, the integration of complex lag elements into decentralized control systems adds a layer of complexity, necessitating further research into adaptive control strategies capable of managing both time delays and system interactions. Future research directions could include the development of advanced machine learning techniques to predict responses in complex lag systems, paving the way for more effective control strategies adaptable to varying environmental conditions.

In summary, the response of complex lag elements is a multifaceted area of study that combines theoretical models, mathematical techniques, and empirical evidence across various fields of application. The literature underscores the importance of accurately modeling and controlling these elements to ensure system stability and performance. As the demand for more sophisticated and responsive systems continues to grow, further exploration into the behavior of complex lag elements will be essential for advancing technology and engineering practices. Given the vital role that these elements play in dynamic systems, ongoing and future research efforts will undoubtedly contribute to a deeper understanding and more effective utilization of complex lag characteristics in practical applications [22], [23] and [24].

3. Step Input Function

Initial conditions for the step function [1], [2], [3], [4] and [5]:

$$\theta_i = 0, \qquad at\ \ t < 0$$
$$\theta_i = k, \qquad at\ \ t \geq 0$$
$$\theta_i = k = 1\ (unit\ step)$$

The standard formula for a system with a complex conversion or transfer factor is:

$$T.0 = \frac{\theta_o}{\theta_i} = \frac{1}{1 + 2\zeta\tau D + \tau^2 D^2}$$

By multiplying inversely (i.e. cross multiplication) the above equation:

$$\theta_o + 2\zeta\tau D\theta_o + \tau^2 D^2\theta_o = \theta_i \ \ \rightarrow (i)$$

The complete solution of the above differential equation is:

$$\theta_o = P.I + C.F$$

Steady State (P.I):

$$\theta_o = \theta_i = \mathrm{k} = 1$$

Unstable state or transient state (C.F):

$$\theta_o = Re^{st}$$
$$\frac{\mathrm{d}\theta_o}{\mathrm{dt}} = D\theta_o = SRe^{st}$$
$$\frac{\mathrm{d}^2\theta_o}{\mathrm{dt}^2} = D^2\theta_o = S^2Re^{st}$$
$$Re^{st} + 2\zeta\tau SRe^{st} + \tau^2S^2Re^{st} = 0$$
$$Re^{st}(1 + 2\zeta\tau S + \tau^2S^2) = 0$$
$$\tau^2\mathrm{S}^2 + 2\zeta\tau S + 1 = 0$$
$$S = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore S = \frac{-2\zeta\tau \pm \sqrt{4\zeta^2\tau^2 - 4\tau^2}}{2\tau^2} = \frac{-2\zeta\tau \pm \sqrt{4\tau^2(\zeta^2 - 1)}}{2\tau^2}$$

$$= \frac{-2\zeta\tau \pm 2\tau\sqrt{\zeta^2 - 1}}{2\tau^2} = \frac{-\zeta}{\tau} \pm \frac{1}{\tau}\sqrt{\zeta^2 - 1}$$

3.1 The First Case

When ζ=0 (Free or Undamped Frequency):

$$S = \frac{-\zeta}{\tau} \pm \frac{1}{\tau}\sqrt{\zeta^2 - 1}$$

Substituting ζ=0,

$$\therefore S = \pm\frac{1}{\tau}\sqrt{-1} = \pm j\frac{1}{\tau}$$

$$\therefore \theta_o = Re^{st} = Re^{\pm j\frac{t}{\tau}}$$

It is known that:

$$\mathrm{e}^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta \rightarrow \text{(ii)}$$

And,

$$\mathrm{e}^{-\mathrm{j}\theta} = \cos\theta - \mathrm{j}\sin\theta \rightarrow \text{(iii)}$$

Adding equations (ii) and (iii), we obtain:

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2}$$

By subtracting equation (iii) from equation (ii), we obtain:

$$\mathrm{j}\sin\theta = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2}$$

$$\therefore \theta_o = R\cos(\theta - \Psi) = R\cos\left(\frac{t}{\tau} - \Psi\right)$$

Complete solution: $\theta_o = P.I + C.F$

$$\therefore \theta_o = 1 + R\cos\left(\frac{t}{\tau} - \Psi\right) \rightarrow (iv)$$

Applying the Boundary Conditions:
When t=o and $\theta_o = 0$ for equation (iv),

$$0 = 1 + R\cos(-\Psi) = 1 + R\cos\Psi$$

$$R\cos\Psi = -1 \qquad \therefore R = \frac{-1}{\cos\Psi}$$

When t=o and $D\theta_o = 0$ for equation (iv),

$$D\theta_o = 0 + \mathrm{R}\left[-\frac{1}{\tau}\sin\left(\frac{t}{\tau} - \Psi\right)\right] = 0$$

$$-\frac{R}{\tau}\sin(-\Psi) = 0$$

$$\therefore \frac{R}{\tau}\sin\Psi = 0 \qquad \therefore \sin\Psi = 0$$

Therefore:

$$\Psi = \sin^{-1} 0 = 0$$

$$\therefore R = \frac{-1}{\cos\Psi} = \frac{-1}{\cos o} = \frac{-1}{1} = -1$$

$$\therefore \theta_o = 1 - \cos\frac{t}{\tau} = 1 - \cos\omega_n t$$

As, $\left(\omega_n = \frac{1}{\tau}\right)$

Figure 1 below shows the first case of the step function when ζ = zero (free or undamped frequency).

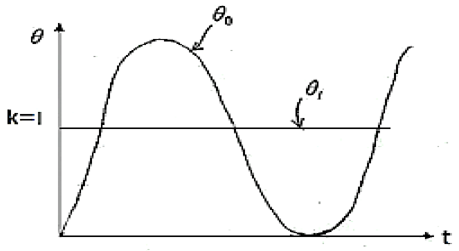


Figure 1: the First Case of the Step Function when ζ = zero (Free or Undamped Frequency)

3.2 The Second Case

When ζ<1 (Under - Damped Oscillatory):

$$s = \frac{-\zeta}{\tau} \pm \frac{1}{\tau}\sqrt{\zeta^2 - 1}$$

Which can be written in the following form:

$$S = \frac{-\zeta}{\tau} \pm j\frac{1}{\tau}\sqrt{1 - \zeta^2}$$

Unsteady state (C.F):

$$\theta_o = Re^{st}$$

$$\theta_o = Re^{\left(-\frac{\zeta}{\tau} \pm j\frac{1}{\tau}\sqrt{1-\zeta^2}\right)t}$$

$$= \mathrm{R}e^{\frac{-\zeta t}{\tau}} \cos\left(\frac{t}{\tau}\sqrt{1 - \zeta^2} - \Psi\right)$$

$\omega_d = \omega_n\sqrt{1 - \zeta^2} = \frac{1}{\tau}\sqrt{1 - \zeta^2}$Damped frequency,

$$\therefore \theta_o = Re^{\frac{-\zeta t}{\tau}} \cos(\omega_d t - \Psi)$$

$$The\ complete\ solution, \theta_o = P.I + C.F$$

$$\theta_o = 1 + Re^{\frac{-\zeta t}{\tau}} \cos(\omega_d t - \Psi)$$

Applying the boundary conditions:

When t=0 and $\theta_o = 0$

The above equation becomes as follows:

$$0 = 1 + R\cos(-\Psi)$$

$$0 = 1 + R\cos\Psi$$

$$R\cos\Psi = -1$$

$$\therefore R = \frac{-1}{\cos\Psi}$$

When t=0 and $D\theta_o = 0$

$$\therefore D\theta_o = 0 + R\left[e^{\frac{-\zeta t}{\tau}} \times -\omega_d \sin(\omega_d t - \Psi) + \cos(\omega_d t - \Psi) \times \frac{-\zeta}{\tau} e^{\frac{-\zeta t}{\tau}}\right]$$

$$0 = R\left[-\omega_d \sin(-\Psi) - \frac{\zeta}{\tau}\cos(-\Psi)\right]$$

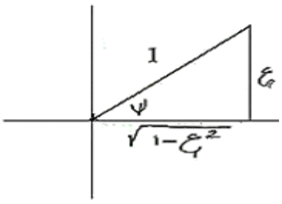
$$0 = \omega_d\, sin\,\Psi - \frac{\zeta}{\tau} cos\,\Psi$$

$$\omega_d \sin\Psi = \frac{\zeta}{\tau}\cos\Psi$$

$$\frac{\sin\Psi}{\cos\Psi} = \tan\Psi = \frac{\zeta}{\tau\omega_d} = \frac{\omega_n\zeta}{\omega_d} = \frac{\omega_n\zeta}{\omega_n\sqrt{1 - \zeta^2}}$$

$$= \frac{\zeta}{\sqrt{1 - \zeta^2}}$$

In addition, using the Pythagorean Theorem as shown in the figure below, we get:



$$tan\,\Psi = \frac{\zeta}{\sqrt{1-\zeta^2}} \quad \therefore \Psi = \sin^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$\cos\Psi = \frac{\sqrt{1-\zeta^2}}{1} = \sqrt{1-\zeta^2}$$

But, $R = \frac{-1}{\cos\Psi} = \frac{-1}{\sqrt{1-\zeta^2}}$

$$\therefore \theta_o = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta t}{\tau}} \cos\left(\frac{t}{\tau}\sqrt{1-\zeta^2} - \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

Figure 2 below shows the Second Case of the Step Function when ζ<1 (under-damped frequency).

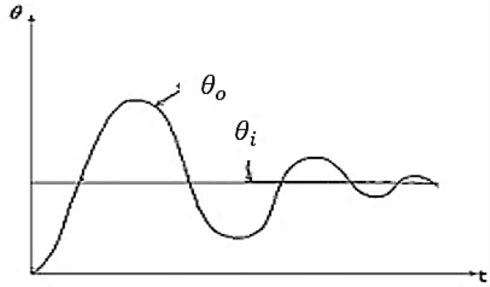


Figure 2: the Second Case of the Step Function when ζ<1 (Under-Damped Frequency)

3.3 The Third Case

When ζ=1 (Critically Damped Frequency):

$$s = \frac{-\zeta}{\tau} \pm \frac{1}{\tau}\sqrt{\zeta^2 - 1}$$

$$S = \frac{-1}{\tau}$$

Unsteady state (C.F):

$$\theta_o = Re^{st} = Re^{\frac{-t}{\tau}}$$

$$The\ complete\ solution, \theta_o = 1 + Re^{\frac{-t}{\tau}}$$

Boundary conditions:

When t=0 and $\theta_o = 0$

$$\theta_o = 1 + Re^{\frac{-t}{\tau}}$$

$$0 = 1 + R \qquad \therefore R = -1$$

$$\therefore \theta_o = 1 - e^{\frac{-t}{\tau}} = 1 - e^{\omega_n t}$$

Figure 3 below shows the third case of the Step Function when ζ = 1 (Critically Damped Frequency).

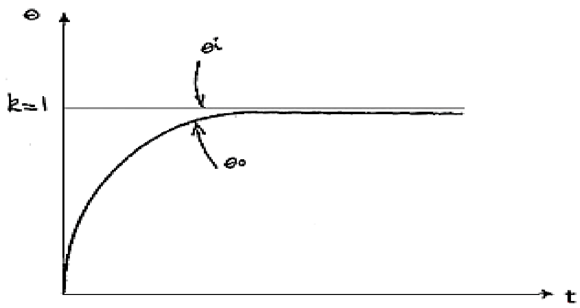


Figure 3: the Third Case of the Step Function when ζ = 1 (Critically Damped Frequency)

3.4 The Fourth Case

When ζ>1 (Over –Damped Frequency (Non-Oscillatory)):

$$S = \frac{-\zeta}{\tau} \pm \frac{1}{\tau}\sqrt{\zeta^2 - 1}$$

Unsteady State (C.F):

$$\theta_o = Re^{st} = Re^{\left(\frac{-\zeta}{\tau} \pm \frac{1}{\tau}\sqrt{\zeta^2-1}\right)t}$$

$$The\ complete\ solution, \theta_o = P.I + C.F$$

$$\theta_o = 1 + Re^{\left(\frac{-\zeta}{\tau} \pm \frac{1}{\tau}\sqrt{\zeta^2-1}\right)t}$$

$$\theta_o = 1 + Re^{\frac{-\zeta}{\tau}t}\left[e^{\pm\frac{t}{\tau}\sqrt{\zeta^2-1}}\right]$$

Figure 4 below shows the Fourth Case of the Step Function when ζ>1 (Over Damped Frequency).

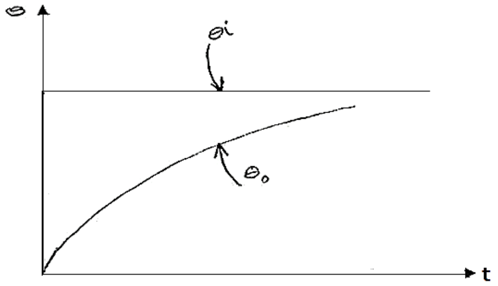


Figure 4: the Fourth Case of the Step Function when ζ>1 (Over Damped Frequency)

3.5 Typical Example of Step Input (refer to references [1] – [6]).

A flywheel driven by an electric motor that is automatically controlled to follow the movement of the hand wheel. The implicit moment of inertia of the flywheel is $150\ kgm^2$ and the motor torque applied to it is $2400\ N.m\ per\ rad$ of misalignment between the flywheel and the handwheel. The viscous friction is equivalent to a torque of $600\ N.m\ rad^{-1}\ s$ if the handwheel is rotated suddenly through 60° when the system is at rest; determine an expression for the angular position of the flywheel with respect to time.

The solution

$I = 150\ kgm^2$ Moment of inertia,

Torsional stiffness, $\lambda = 2400\ N.m/rad$

$C = 600\ N\ .ms/rad$ Coefficient of viscous damping,

$\theta_i = 60° = \frac{60\times\pi}{180} = \frac{\pi}{3}$ rad Handwheel displacement,

$$\theta_o(t) = ?$$

Figure 5 below shows a flywheel driven by an electric motor that is automatically controlled to follow the movement of the handwheel.

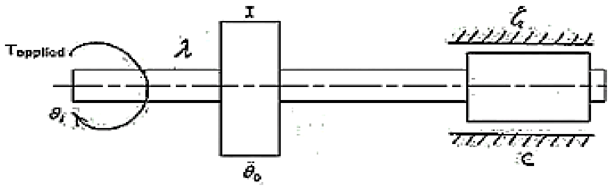


Figure 5: a Flywheel Driven by an Electric Motor

Equation of motion:

$$\lambda(\theta_i - \theta_o) - C\theta_o^{\circ} = I\theta_o^{\circ\circ}$$
$$\lambda\theta_i - \lambda\theta_o - \mathrm{CD}\theta_o = ID^2\theta_o$$
$$\lambda\theta_i = \lambda\theta_o + \mathrm{CD}\theta_o + ID^2\theta_o$$

$T.0 = \frac{\lambda}{\lambda + CD + ID^2}$ Transfer function or operator,

By dividing the numerator and denominator by λ:

$$\frac{\theta_o}{\theta_i} = \frac{1}{1 + \frac{C}{\lambda}D + \frac{I}{\lambda}D^2}$$

Which is analogous to the standard form of complex delay or lag,

$$\frac{1}{1 + 2\zeta\tau D + \tau^2 D^2}$$

$$\therefore 2\zeta\tau = \frac{C}{\lambda} \text{ ، } \quad 2\zeta\tau = \frac{600}{2400} = 0.25 \ \rightarrow (i)$$

$\tau^2 = \frac{I}{\lambda} = \frac{150}{2400}$ ، $\therefore \tau = \sqrt{\frac{150}{2400}} = 0.25 \sec \rightarrow (ii)$ Also,

Substituting the value of τ from equation (ii) into equation (i),

$$2 \times 0.25\zeta = 0{,}25$$
$$\therefore \zeta = \frac{0.25}{0.5} = 0.5$$

Since ζ<1, the frequency (vibration) is under damped.

$\theta_o = \theta_i \left(1 + Re^{\frac{-\zeta t}{\tau}} \cos(\omega_d t - \Psi)\right)$ ، Step function response

$$\theta_o = \frac{\pi}{3}\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta t}{\tau}} \cos\left(\frac{t}{\tau}\sqrt{1-\zeta^2} - \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)$$

$$\theta_o = \frac{\pi}{3}\left(1 - \frac{1}{\sqrt{1-0.5^2}} e^{(-0.5/0.25)t} \cos\left(\frac{t}{0.25}\sqrt{1-0.5^2} - \tan^{-1}\frac{0.5}{\sqrt{1-0.5^2}}\right)\right)$$

$$\therefore \theta_o = \frac{\pi}{3}[1 - 1.155e^{-2t}\cos(3.46t - 0.58)]$$

3.6 Typical Example of Ramp Input (refer to references [1] – [5]).

A position control system that controls the angular displacement of the load by applying a torque directly proportional to the error (i.e. the difference between input and output). The moment of inertia of the load is 340 kgm^2 and the damping coefficient is equal to 8000 N.m/(rad/s) when the input is 10 deg/s. The stable error or stabilizing error equals 0.25°, find:

a] Control constant k.

b] Damping ratio ζ.

c] Damped natural frequencyω_d.

d] Undamped natural frequencyω_n.

The solution:

Figure 6 below shows a position control system that controls the angular displacement of the load.

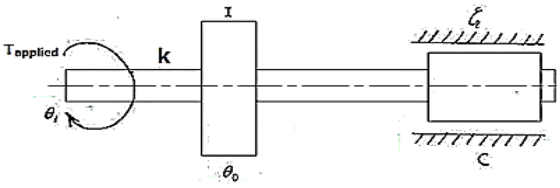


Figure 6: a Position Control System

Equation of motion:

$$k(\theta_i - \theta_o) - CD\theta_o = ID^2\theta_o$$
$$k\theta_i - k\theta_o - CD\theta_o = ID^2\theta_o$$
$$k\theta_i = k\theta_o + CD\theta_o + ID^2\theta_o$$
$$k\theta_i = \theta_o[k + CD + ID^2]$$

$T.0 = \frac{\theta_o}{\theta_i} = \frac{k}{k+CD+ID^2}$ Transfer function or operator,

Dividing the numerator and denominator by k, we get:

$$\frac{\theta_o}{\theta_i} = \frac{1}{1 + \frac{C}{K}D + \frac{I}{K}D^2}$$

Which is analogous to the standard form of complex lag $1/(1+2\zeta\tau D+\tau^2 D^2)$.

Problem given data:

$C = 8000N.\frac{m}{\left(\frac{rad}{s}\right)}, I = 340kgm^2$ Coefficient of damping,

$$\omega_i = 10^\circ/s = \frac{10^\circ \times \pi}{180} = 0.1745\text{rad/s}$$

$\epsilon_{ss} = 0.25° = 0.25 \times \frac{\pi}{180} = 0.00436rad$ Steady state error,

Response to ramp input:

$$\theta_i = \omega t$$

Steady state (P.I):

$$D\theta_o = \omega$$
$$D^2\theta_0 = 0$$
$$\theta_o + 2\zeta\tau D\theta_o + \tau^2 D^2\theta_o = \theta_i$$
$$\omega t + Q + 2\zeta\tau\omega + 0 = \omega t$$
$$\therefore Q = -2\zeta\tau\omega$$

$\theta_o = \omega t - 2\zeta\tau\omega$ Steady state response,

$$\theta_i = \omega t$$

$\epsilon_{ss} = \theta_i - \theta_o = \omega t - \omega t + 2\zeta\tau\omega = 0.0043$ Steady state error,

$$\epsilon_{ss} = 0.00436 = 2\zeta\tau\omega$$

$but, \omega = 0.1745rad/s$

$$\therefore 0.00436 = 2\zeta\tau \times 0.1745$$
$$\therefore 2\zeta\tau = \frac{0.00436}{0.1745} = 0.025 \rightarrow \text{(i)}$$

a] Control constant, k:

$$Control\ constant, k = \frac{C}{2\zeta\tau} = \frac{8000}{0.025} = 320{,}000N.m/rad$$
$$= 320KN.m/rad$$

b] Damping ratio, ζ:

$$\tau^2 = \frac{I}{K} \quad \therefore \tau = \sqrt{\frac{I}{K}} = \sqrt{\frac{340}{320 \times 10^3}} = 0.0326s/rad$$

$$2\zeta \times 0.0326 = \frac{8000}{320 \times 10^3}$$

$$\therefore \text{Damping ratio}, \zeta = \frac{8000}{2 \times 0.0326 \times 320 \times 10^3} = 0.383$$

c] Damped natural frequency, ω_d:

Damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\omega_n = \frac{1}{\tau} = \frac{1}{0.0326} = 30.7 rad/s$$

$$\therefore \omega_d = 30.7\sqrt{1-0.383^2} = 28.36\, rad/s$$

d] Undamped natural frequency, ω_n:

Undamped natural frequency, $\omega_n = 30.7\, rad/s$

3.7 Typical Example of Adding the First Differential Derivative of the Error ([1] – [6])

The base mass of the anti-aircraft gun is 7.5 mg and the radius of gyration around its axis of rotation is 0.76 m. A direct control device plus an error differential is used for the angular displacement of the gun from a distance. The damping coefficient around the cannon axis is 10.9 KN.m/(rad/s), when the cannon is moving at a maximum speed of 25 rev/min the lag angle should be no more than 2° and the damping ratio is 0.5, find:

a] Proportional control constant and differential error control constant.

b] The time of the cycle when applying a sudden input.

c] The servo motor power when moving the load at maximum speed.

The solution:

$$m = 7.5 \times 10^3\, kg$$

The radius of gyration motion or the radius of the moment of inertia,

$$k_G = 0.76\, m$$

$I = mK_G{}^2 = 7.5 \times 10^3 \times 0.76^2 = 4332kgm^2$ Moment of inertia,

Damping coefficient around the gun axis, $C = 10.9 \times 10^3 N.\frac{m}{\left(\frac{rad}{s}\right)}$

$$Max.Speed\ in \frac{rev}{min}, N_{max} = 25rev/min$$

$\omega_{max} = \frac{25\times2\pi}{60} = 2.618 rad/s$ Max. Speed in rad/s,

$\epsilon_{ss} = \frac{2°\times\pi}{180} = 0.035rad$ ، Steady-state error or stabilization error

$\zeta = 0.5$ ، Damping ratio

a] the control constant k and the error differential control constant k_1.

Figure 7 below shows the proportional control constant for the base mass of an anti-aircraft gun without adding the first differential derivative of the error.

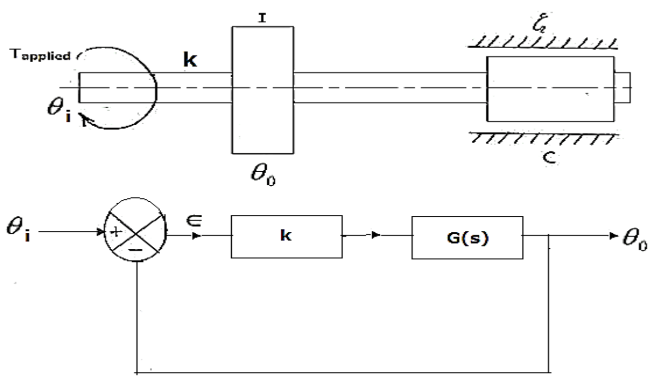


Figure 7: the Proportional Control Constant for the Base Mass of an Anti-aircraft Gun without adding the First Differential Derivative of the Error

Response to ramp input,

$$\theta_i = \omega t$$

Steady State (P.I):

$$\theta_o = \omega t + Q$$
$$D\theta_o = \omega$$
$$D^2\theta_o = 0$$

Motion equation:

$$k(\theta_i - \theta_o) - CD\theta_o = ID^2\theta_o$$
$$k\theta_i - k\theta_o - CD\theta_o = ID^2\theta_o$$
$$k\theta_i = k\theta_o + CD\theta_o + ID^2\theta_o$$
$$= \theta_o[k + CD + ID^2]$$
$$\frac{\theta_o}{\theta_i} = \frac{k}{k + CD + ID^2}$$
$$\frac{\theta_o}{\theta_i} = \frac{1}{1 + \frac{C}{K}D + \frac{I}{K}D^2}$$

Which is analogous to the standard formula for complex lag, $1/(1+2\zeta\tau D+\tau^2 D^2)$.

$$\theta_o + \frac{c}{k}D\theta_o + \frac{1}{k}D^2\theta_o = \theta_i$$
$$\omega t + Q + \frac{c}{k}\omega + \frac{I}{k} \times 0 = \omega t$$
$$\omega t + Q + \frac{c}{k}\omega = \omega t$$
$$\therefore Q = -\frac{c}{k}\omega$$

But, $\theta_o = \omega t + Q$

$$\therefore \theta_O \omega t - \frac{c}{k}\omega$$
$$\theta_i = \omega t$$

$\in_{ss} = \theta_i - \theta_o$ Steady state error,

$$\in_{SS} = \omega t - \left(\omega t - \frac{c}{k}\omega\right)$$
$$\therefore \in_{SS} = \omega t - \omega t + \frac{c}{k}\omega = \frac{c}{k}\omega$$

In symmetry of the transfer function with the standard complex lag or delay form,

$$\frac{c}{k} = 2\zeta\tau \rightarrow (i)$$
$$\frac{I}{k} = \tau^2 \rightarrow (ii)$$
$$\text{As}, \in_{ss} = \frac{C}{K}\omega \rightarrow (iii)$$

By substituting into equation (iii):

$$0.035 = \frac{10.9 \times 10^3}{k} \times 2.618$$
$$\therefore k = \frac{10.9 \times 10^3 \times\times 2.618}{0.035} = 815320 N.m/rad$$
$$= 815.32 KN.m/rad$$

From equation (ii):

$$\frac{I}{k} = \tau^2 \quad , \quad \frac{4332}{815.32 \times 10^3} = \tau^2$$
$$\therefore \tau = 0.0729 sec/rad$$

By adding the first differential derivative of the error,

Motion equation:

$$k(\in + k_1 D \in) = CD\theta_o + ID^2\theta_o$$

$$\in = \theta_i - \theta_o$$
$$k[(\theta_i - \theta_o) + k_1 D(\theta_i - \theta_o)] = CD\theta_o + ID^2\theta_o$$
$$k\theta_i - k\theta_o + kk_1 D\theta_i - kk_1 D\theta_o = CD\theta_o + ID^2\theta_o$$
$$k\theta_i + kk_1 D\theta_i = k\theta_o + kk_1 D\theta_o + CD\theta_o + ID^2\theta_o$$
$$k\theta_i(1 + k_1 D) = \theta_o[k + kk_1 D + CD + ID^2]$$
$$k\theta_i(1 + k_1 D) = k\theta_o\left[1 + k_1 D + \frac{C}{K}D + \frac{I}{K}D^2\right]$$
$$T.0 = \frac{\theta_o}{\theta_i} = \frac{1 + k_1 D}{1 + \left(k_1 + \frac{C}{K}\right)D + \frac{I}{K}D^2}$$
$$\frac{\theta_o}{\theta_i} = \frac{1 + k_1 D}{1 + \left(\frac{kk_1 + c}{k}\right)D + \frac{I}{K}D^2} \quad \rightarrow (iv)$$

Figure 8 below shows the control constant of the differential error when adding the first differential derivative of the error.

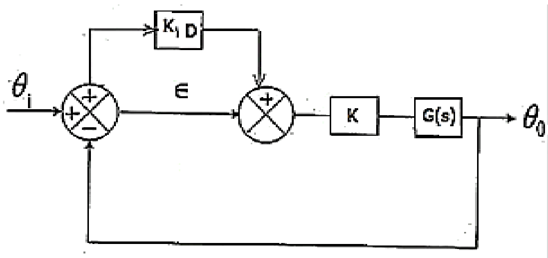


Figure 8: the Control Constant of the Differential Error when adding the First Differential Derivative of the Error

Which is analogous to the standard form:

$$\frac{1}{1 + 2\zeta\tau D + \tau^2 D^2}$$
$$\theta_o + \left(\frac{kk_1 + c}{k}\right)D\theta_o + \frac{I}{K}D^2\theta_o = \theta_i + k_1 D\theta_i$$

Ramp input response:

$$\theta_i = \omega t$$
$$D\theta_i = \omega$$

Steady State (P.I):

$$\theta_o = \omega t + Q$$
$$D\theta_o = \omega$$
$$D^2\theta_o = 0$$
$$\omega t + Q + \left(\frac{kk_1 + c}{k}\right)\omega + 0 = \omega t + k_1\omega$$
$$Q = k_1\omega - \left(\frac{kk_1 + c}{k}\right)\omega = \left[k_1 - \left(\frac{kk_1 + c}{k}\right)\right]\omega$$
$$\theta_o = \omega t + \left[k_1 - \left(\frac{kk_1 + c}{k}\right)\right]\omega$$
$$\epsilon_{ss} = \theta_i - \theta_o$$
$$\epsilon_{ss} = \omega t - \omega t - \left[k_1 - \left(\frac{kk_1 + c}{k}\right)\right]\omega$$
$$\epsilon_{ss} = -\left[k_1 - \left(\frac{kk_1 + c}{k}\right)\right]\omega = \left[\frac{kk_1 + c}{k} - k_1\right]\omega$$

From equation (iv),

$$\frac{kk_1 + c}{k} = 2\zeta\tau \rightarrow (v)$$

$$\frac{I}{K} = \tau^2 \rightarrow (vi)$$

From equation (v),

$$\frac{815.32 \times 10^3 k_1 + 10.9 \times 10^3}{815.32 \times 10^3} = 2 \times 0.5 \times 0.0729$$

$$\Rightarrow \therefore k_1 = 0.0595 sec$$

b] Cycle time when receiving a sudden input:

$t_p = \frac{2\pi}{\omega_d}$ ، Cycle time

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$= \frac{1}{\tau} \sqrt{1 - \zeta^2}$$

$$\therefore \omega_d = \frac{1}{0.0729} \sqrt{1 - 0.5^2} = 11.88 rad/s$$

$$\therefore t_p = \frac{2\pi}{11.88} = 0.529 sec$$

c] Power of the servomotor when moving the load at maximum speed:

$P = T\omega$ ، Power

$T = C\omega$ ، Torque

$$\therefore P = C\omega^2 = 10.9 \times 10^3 \times 2.618^2 = 74707.8 W$$

$$= 74.71 KW$$

4. Conclusions

Complex lag elements are essential in system dynamics and control theory, combining lag and lead characteristics to analyze systems impacted by delays and phase shifts. This literature review examines their theoretical foundations, mathematical representations, and practical applications, especially in control systems and signal processing. Complex lag elements, which incorporate time delays and phase shifts, are represented using Laplace domain transfer functions. Researchers emphasize their importance for stability and control design.

Mathematical modeling often employs techniques such as Padé approximations, frequency response analysis, and root locus techniques to optimize control strategies and enhance system stability in the presence of uncertainties. Empirical studies highlight their application across fields, including automotive and aerospace systems, where managing response time is critical for operational effectiveness and safety.

However, challenges remain in accurately modeling time delays, particularly in nonlinear systems, and integrating these elements into decentralized control systems. Future research may focus on advanced machine learning methods to better predict behaviors in complex lag systems.

Additionally, the text compares first-order and second-order systems, noting that second-order systems exhibit more complex dynamics, transient responses, and frequency behaviors, requiring advanced control techniques. Derivative Error Compensation is mentioned as a method to improve system performance by enhancing damping and reducing settling time, though it may also amplify measurement noise if not properly filtered. Overall, further exploration of complex lag elements is crucial for advancing technology in dynamic systems.

References

[1] R.K. Rajput (Author), Mechanical Measurements and Instrumentation (Including Metrology and Control Systems) Paperback – 1 January 2013.

[2] Osama Mohammed Elmardi Suleiman Khayal (Author), Mechanical Instrumentation and Measurement Devices: Solved Examples and Additional Further Problems, ASIN **:** B0CK3M5GTZ Publisher **:** LAP LAMBERT Academic Publishing (September 7, 2023), Language **:** English, Paperback **:** 220 pages, ISBN-10 **:** 6206782549, ISBN-13: 978-6206782544.

[3] Osama Mohammed Elmardi Suleiman Khayal (Author), Automation and Control of Exponential and Complex Lag Systems Paperback – 17 September 2023, ASIN **:** B0CKKPY9HC, Publisher **:** LAP LAMBERT Academic Publishing (17 September 2023); LAP LAMBERT Academic Publishing, Language **:** English, Paperback **:** 356 pages, ISBN-10 **:** 6206784983, ISBN-13: 978-6206784982.

[4] John Hannah, Richmond Courtney Stephens, " Mechanics of Machines: Elementary Theory and Examples “, Volume 1, (1984).

[5] William S. Levine (Editor), The Control Systems Handbook: Control System Advanced Methods, Second Edition (The Electrical Engineering Handbook) 2nd Edition, Publisher : CRC Press; 2nd edition (December 8, 2010), Language : English, Hardcover : 1798 pages, ISBN-10 : 1420073648, ISBN-13: 978-1420073645.

[6] Bakshi, Varsha U.., Bakshi, Uday A. Control System Engineering. India: Amazon Digital Services LLC - KDP Print US, 2020.1. Frank, S.A. (2018). Time Delays. In: Control Theory Tutorial. Springer Briefs in Applied Sciences and Technology. Springer, Cham.

[7] Ning Wang, John Hagedoorn, The lag structure of the relationship between patenting and internal R&D revisited**,** Research Policy**,** Volume 43, Issue 8**,** 2014 **,**Pages 1275-1285**,** ISSN 0048-7333**,** https://doi.org/10.1016/j.respol.2014.03.010.

[8] Aubrun, J.N., K.R. Lorell, T.S. Mast, and J.E. Nelson (1987). “Dynamic analysis of the actively controlled segmented mirror of the W.M. Keck ten-meter telescope,” IEEE Control Syst. Mag., vol. 7, no. 6, pp. 3-10.

[9] Aubrun, J.N., K.R. Lorell, T.W. Havas, and W.C. Henninger (1988). “Performance analysis of the segmented alignment control system for the ten-meter telescope,” Automatica, vol. 24, pp. 437-454.

[10] Bensoussan, D. (1984). “Sensitivity reduction in single-input single-output systems,” Int. J. Control, vol. 39, pp. 321-335.

[11] Boyd, S.P., V. Balakrishnan, C.H. Barratt, N.M. Khraishi, X. Li, D.G. Meyer, and S.A. Norman (1988). “A new CAD method and associated architectures for linear controllers,” IEEE Trans. Auto. Control, vol. AC-33, pp. 268-283.

[12] Zames, G. (1981). “Feedback and optimal sensitivity: model reference transformations, multiplicative semi norms, and approximate inverses,” IEEE Trans. Auto. Control, vol. AC-26, pp. 301-320.

[13] Zames, G. and B.A. Francis (1983). “Feedback, minimax sensitivity, and optimal robustness,” IEEE Trans. Auto. Control, vol. AC-28, pp. 585-601.

[14] John Doyle, Bruce Francis, Allen Tannenbaum. Feedback Control Theory, Macmillan Publishing Co., 1990.

[15] Franklin, G & Powell, J.D. & Emami-Naeini, Abbas. (1994). Feedback Control Of Dynamic Systems.

[16] Wallis, B.A., Watt, K., Franklin, R.C., Nixon, J.W., et al. (2015) Where Children and Adolescents Drown in Queensland: A Population-Based Study. BMJ Open, 5, 26.

[17] Pekař, L., Song, M., Gao, Q. (2024). On the Relation Between the Smith Predictor and Algebraic Control Approach for Time Delay Systems: A Case Study. In: Machado, J., et al. Innovations in Mechatronics Engineering III. icieng 2024. Lecture Notes in Mechanical Engineering. Springer, Cham.

[18] Mehta, Utkal & Kaya, Ibrahim. (2017). Smith predictor with sliding mode control for processes with large dead times. Journal of Electrical Engineering. 68. 463–46. 10.1515/jee-2017-0081.

[19] Åström, K. J., Wittenmark, B. (2011). Computer-Controlled Systems: Theory and Design. United States: Dover Publications.

[20] Saha, Indranil & Baruah, Sanjoy & Majumdar, Rupak. (2015). Dynamic scheduling for networked control systems. Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control, HSCC 2015. 98-107. 10.1145/2728606.2728636.

[21] van Horssen, E. P., Antunes, D., & Heemels, M. (2019). Switched LQG control for linear systems with multiple.

[22] Al-Areqi, S., Görges, D., Reimann, S., & Liu, S. (2013). Event-based control and scheduling codesign of networked embedded control systems. In American control conference (pp. 5299–5304).

[23] Albers, R., Suijs, E., & de With, P. H. N. (2009). Resource prediction and quality control for parallel execution of heterogeneous medical imaging tasks. In 2009 16th IEEE international conference on image processing (pp. 2317–2320).

[24] Antunes, D., & Heemels, W. P. M. H. (2014). Rollout event-triggered control: Beyond periodic control performance. Institute of Electrical and Electronics Engineers. Transactions on Automatic Control, 59, 3296–3311.