Transient State Heat Conduction: A Review

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Abstract

The research provides a comprehensive understanding of heat transfer for engineering students, covering various topics and emphasizing key theoretical principles. It's a useful resource for degree programs in mechanical, production, and chemical engineering, as well as interdisciplinary studies. The study explores transient heat conduction, using mathematical foundations, analytical methods to investigate how temperature distribution changes over time. The outcomes highlight the impact of material properties, geometry, and initial conditions on heat conduction behavior, offering valuable insights for applications in engineering, material science, and thermal management.

Keywords: Unsteady Conduction; Lumped Capacitance; Time Constant and Response of Temperature; Semi – Infinite Solids, Finite Solid; Periodic Variation; Temperature Distribution**.**

1. Introduction

From the study of thermodynamics, you have learned that energy can be transferred through interactions between a system and its surroundings, which are referred to as work and heat. However, thermodynamics primarily focuses on the final states of processes during these interactions, offering little insight into the nature of the interactions or the rates at which they occur. The goal of this research is to enhance thermodynamic analysis by investigating transient conduction heat transfer and developing relationships to calculate various variables in lumped capacitance theory.

In previous studies of conduction, we progressively addressed more complex scenarios. We started with the straightforward case of one-dimensional, steady-state conduction without internal generation and later explored more realistic conditions that include multidimensional and generation effects. However, we have not yet examined scenarios in which conditions vary over time.

We now understand that many heat transfer problems are time-dependent. These unsteady, or transient, issues typically emerge when the boundary conditions of a system are modified. For instance, changing the surface temperature of a system will cause temperature adjustments throughout the entire system until a steady-state temperature distribution is established. Consider a hot metal billet taken from a furnace and exposed to a cool air stream. Energy is transferred away from its surface to the surroundings through convection and radiation. Additionally, energy is conducted from the interior of the metal to its surface, resulting in a decrease in temperature at every point within the billet until steady state is achieved. The final properties of the metal will be significantly influenced by the temperature-history profile that arises from this heat transfer.

Effectively controlling heat transfer is essential for fabricating new materials with improved characteristics.

Our research aims to establish methods for determining the time-dependent temperature distribution within a solid during a transient process, as well as for assessing heat transfer between the solid and its surroundings. The specific approach adopted depends on the assumptions that can be made regarding the process. For instance, if the temperature gradients within the solid can be disregarded, a relatively straightforward approach known as the lumped capacitance method or negligible internal resistance theory can be employed to analyze the change in temperature over time.

In cases where temperature gradients cannot be ignored, yet heat transfer within the solid is one-dimensional, exact solutions to the heat equation can be utilized to evaluate temperature dependence on both spatial and temporal variables. These solutions apply to both finite and infinite solids, and we also investigate the response of a semi-infinite solid to periodic heating at its surface.

Transient conduction is crucial in many engineering applications; for instance, when an engine is started, a certain amount of time must elapse before steady state is reached, and the processes during this transitional time can have adverse effects. Similarly, when quenching metal, understanding the temperature-time history is vital. One particular scenario we will analyze involves cases where the internal or conductive resistance of the body is minimal and negligible compared to the external or convective resistance.

This configuration is known as a lumped capacity or capacitance system and also referred to as a negligible internal resistance system because the internal resistance is low, conductivity is high, and the conduction heat flow rate is substantial, leading to negligible temperature variation throughout the body. The extent of internal resistance is quantified by the Biot number (Bi), which is the ratio of conductive resistance to convective resistance.

$$i.e.\ Bi = \frac{hl}{k}$$

When $Bi \ll 0.1$ the system can be assumed to be of lumped capacity (i.e. at Bi = 0.1 the error is less than 5% and as Bi becomes less the accuracy increases) [1] – [31].

2. Overview

During any period in which temperatures changes in time at any place within an object, the mode of thermal energy flow is termed transient conduction. Another term is "non-steady-state" conduction, referring to the time-dependence of temperature fields in an object. Non-steady-state situations appear after an imposed change in temperature at a boundary of an object. They may also occur with temperature changes inside an object, as a result of a new source or sink of heat suddenly introduced within an object, causing temperatures near the source or sink to change in time.

When a new perturbation of temperature of this type happens, temperatures within the system change in time toward a new equilibrium with the new conditions, provided that these do not change. After equilibrium, heat flow into the system once again equals the heat flow out, and temperatures at each point inside the system no longer change. Once this happens, transient conduction is ended, although steady-state conduction may continue if heat flow continues.

If changes in external temperatures or internal heat generation changes are too rapid for the equilibrium of temperatures in space to take place, then the system never reaches a state of unchanging temperature distribution in time, and the system remains in a transient state.

An example of a new source of heat "turning on" within an object, causing transient conduction, is an engine starting in an automobile. In this case, the transient thermal conduction phase for the entire machine is over, and the steady-state phase appears, as soon as the engine reaches steady-state operating temperature. In this state of steady-state equilibrium, temperatures vary greatly from the engine cylinders to other parts of the automobile, but at no point in space within the automobile does temperature increase or decrease. After establishing this state, the transient conduction phase of heat transfer is over.

New external conditions also cause this process: for example, the copper bar in the example steady-state conduction experiences transient conduction as soon as one end is subjected to a different temperature from the other. Over time, the field of temperatures inside the bar reaches a new steady-state, in which a constant temperature gradient along the bar is finally set up, and this gradient then stays constant in time. Typically, such a new steady-state gradient is approached exponentially with time after a new temperature-or-heat source or sink, has been introduced. When a "transient conduction" phase is over, heat flow may continue at high power, so long as temperatures do not change.

An example of transient conduction that does not end with steady-state conduction, but rather no conduction, occurs when a hot copper ball is dropped into oil at a low temperature. Here, the temperature field within the object begins to change as a function of time, as the heat is removed from the metal, and the interest lies in analyzing this spatial change of temperature within the object over time until all gradients disappear entirely (the ball has reached the same temperature as the oil). Mathematically, this condition is also approached exponentially; in theory, it takes infinite time, but in practice, it is over, for all intents and purposes, in a much shorter period. At the end of this process with no heat sink but the internal parts of the ball (which are finite), there is no steady-state heat conduction to reach. Such a state never occurs in this situation, but rather the end of the process is when there is no heat conduction at all.

The analysis of non-steady-state conduction systems is more complex than that of steady-state systems. If the conducting body has a simple shape, then exact analytical mathematical expressions and solutions may be possible (see heat equation for the analytical approach). However, most often, because of complicated shapes with varying thermal conductivities within the shape (i.e., most complex objects, mechanisms or machines in engineering) often the application of approximate theories is required, and/or numerical analysis by computer. One popular graphical method involves the use of Heisler Charts.

Occasionally, transient conduction problems may be considerably simplified if regions of the object being heated or cooled can be identified, for which thermal conductivity is very much greater than that for heat paths leading into the region. In this case, the region with high conductivity can often be treated in the lumped capacitance model, as a "lump" of material with a simple thermal capacitance consisting of its aggregate heat capacity. Such regions warm or cool, but show no significant temperature variation across their extent, during the process (as compared to the rest of the system). This is due to their far higher conductance. During transient conduction, therefore, the temperature across their conductive regions changes uniformly in space, and as a simple exponential in time. An example of such systems is those that follow Newton's law of cooling during transient cooling (or the reverse during heating). The equivalent thermal circuit consists of a simple capacitor in series with a resistor. In such cases, the remainder of the system with a high thermal resistance (comparatively low conductivity) plays the role of the resistor in the circuit, [32], [33], [34], [35], [36] and [37].

3. Definition of Lumped Capacity or Capacitance System

Lumped Capacity or Capacitance System is the system where the internal or conductive resistance of a body is very small or negligible compared to the external or convective resistance.
Biot number (Bi): is the ratio between the conductive and convective resistance.

$$Bi = \frac{hl}{k}$$

$$Bi = \frac{conduction\ resistance}{convection\ resistance} = \frac{x}{kA} / \frac{1}{hA} = \frac{x}{kA} \times \frac{hA}{1} = \frac{hx}{k}$$

Where x is the characteristic linear dimension and can be written as l, and h is the heat transfer coefficient by convection and k is the thermal conductivity. When $Bi \ll 0.1$, the system is assumed to be of lumped capacity [1], [3], [6], [38] and [39].

4. Characteristic Linear Dimensions of Different Geometries

The characteristic linear dimension of a body, $\mathrm{L} = \frac{\mathrm{V}}{\mathrm{A}_s} = \frac{\text{volume of the body}}{\text{surface area of the body}}$

Characteristic linear dimension of a plane surface, $\mathrm{L} = \frac{\mathrm{t}}{2}$

Characteristic linear dimension of a cylinder, $\mathrm{L} = \frac{\mathrm{r}}{2}$

Characteristic linear dimension of a sphere (ball), $\mathrm{L} = \frac{\mathrm{r}}{3}$

Characteristic linear dimension of a cube, $\mathrm{L} = \frac{\mathrm{a}}{6}$

Where: t is the plate thickness, r is the radius of a cylinder or sphere, and a is the length side of a cube. The derivations of the above characteristic lengths are given below:

i) The characteristic length of plane surface, $L = \frac{t}{2}$. Figure 1 below shows the dimensions of a plane surface.

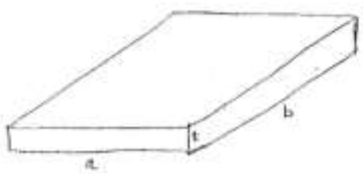

Figure 1: The Dimensions of a Plane Surface

$$V = abt$$

$$A_s = 2at + 2bt + 2ab$$

Since, t is very small; therefore, it can be neglected.

$$A_s = 2ab$$

$$L = \frac{V}{A_s} = \frac{abt}{2ab} = \frac{t}{2}$$

ii) The characteristic length of cylinder,

$$L = \frac{V}{A_s} = \frac{r}{2}$$

$$V = \pi r^2 L$$

$$A_s = 2\pi r L$$

$$\therefore\ L = \frac{V}{A_s} = \frac{\pi r^2 L}{2\pi r L} = \frac{r}{2}$$

iii) The characteristic length of a sphere (ball),

$$L = \frac{r}{3}$$

$$v = \frac{4}{3}\pi r^3$$

$$A_s = 4\pi r^2$$

$$\therefore\ L = \frac{V}{A_s} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$$

iv) The characteristic length of a cube. Figure 2 below shows the dimensions of a cube.

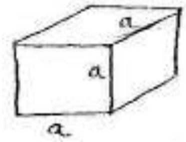

Figure 2: The Dimensions of a Cube

$$L = \frac{a}{6}$$

$$V = a^3$$

$$A_s = 6a^2$$

$$L = \frac{V}{A_s} = \frac{a^3}{6a^2} = \frac{a}{6}$$

5. Derivation of Equations of Lumped Capacitance System

Consider a hot body of an arbitrary shape as shown in Figure 3 below.

Energy balance at any instant requires that: The rate of loss of internal energy of the body must be equal to the rate of convection from the body to the surrounding fluid.

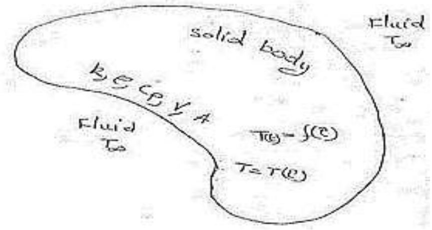

Figure 3: Hot Body of an Arbitrary Shape

$$\text{q} = -\rho V c_p \frac{dT(t)}{d\tau} = hA_s(T(t) - T_\infty) \qquad (1)$$

Put $(T(t) - T_\infty) = \theta$

And,

$$\frac{dT(t)}{d\tau} = \frac{d\theta}{d\tau}$$

$$\therefore -\rho V c_P \frac{d\theta}{d\tau} = h\, A_s \theta \qquad (2)$$

If the temperature of the body at time $\tau = 0$ is equal to T_o

$$\therefore \theta_o = T_o - T_\infty$$

$$-\rho V c_P \frac{d\theta}{\theta} = hA_s\, d\tau \qquad (3)$$

By integrating the above equation (3) we obtain the following equation:

$$-\rho V c_P \int_{\theta_o}^{\theta} \frac{d\theta}{\theta} = \int_{\tau=0}^{\tau=\tau} hA_s\, d\tau$$

$$-\rho V \ln\frac{\theta}{\theta_o} = hA_s\, \tau$$

$$\ln\frac{\theta}{\theta_o} = -\frac{hA_s\, \tau}{\rho V c_P}$$

$$\therefore \ \frac{\theta}{\theta_o} = e^{\frac{-hA_s \tau}{\rho V c_P}} \qquad (4)$$

$$\frac{hA_s}{\rho V c_P} \cdot \tau \text{ can be written as } \frac{h\,V}{kA_s} \cdot \frac{{A_s}^2 k}{V^2 \rho c_P} \tau \qquad (5)$$

$$\frac{k}{\rho c_P L^2} \cdot \tau = \text{Fourier number Fo (dimensionless Quantity)} \qquad (6)$$

And,

$$\frac{h\,L}{k} = Bi \qquad (7)$$

$$\therefore \ \frac{hA_s}{\rho V c_P} \cdot \tau = Bi \times Fo \qquad (8)$$

$$\therefore \ \frac{\theta}{\theta_o} = \frac{T(t) - T_\infty}{T_o - T_\infty} = e^{-Bi \times Fo} \qquad (9)$$

The instantaneous heat transfer rate $q'(\tau)$ is given by the following equation:

$$q'(\tau) = hA_s\theta = hA_s\theta_o e^{-Bi \times Fo} \qquad (10)$$

At time $\tau = 0$,

$$q'(\tau) = hA_s\theta_o \qquad (11)$$

The total heat transfer rate from $\tau = 0$ to $\tau = \tau$ is given by the following equation:

$$Q(t) = \int_{\tau=0}^{\tau=\tau} q'(\tau) = \int_0^{\tau} hA_s\theta_o e^{-Bi \times Fo} \qquad (12)$$

From equation (8) $\rightarrow Bi \times Fo = \frac{hA_s}{\rho V c_P} \cdot \tau$ and substitute in equation (12), the following equation is obtained [1], [3], [6], and [40]:

$$Q(t) = hA_s\theta_o \int_0^{\tau} e^{\frac{-hA_s}{\rho V c_P} \cdot \tau} = hA_s\theta_o \left[\frac{e^{\frac{-hA_s}{\rho V c_P} \cdot \tau}}{\frac{-hA_s}{\rho V c_P}} \right]_o$$

$$= hA_s\theta_o \, e^{\frac{-hA_s}{\rho V c_P} \cdot \tau} \times \frac{-\rho V c_P}{hA_s} = -hA_s\theta_o \frac{\rho V c_P}{hA_s} \left[e^{\frac{-hA_s}{\rho V c_P} \cdot \tau} \right]_o^{\tau}$$

$$= -hA_s\theta_o \frac{\rho V c_P}{hA_s} e^{\frac{-hA_s}{\rho V c_P} \cdot \tau} - \left\{ -hA_s\theta_o \frac{\rho V c_P}{hA_s} \right\}$$

$$= -hA_s\theta_o \frac{\rho V c_P}{hA_s} e^{\frac{-hA_s}{\rho V c_P} \cdot \tau} + hA_s\theta_o \frac{\rho V c_P}{hA_s}$$

$$= hA_s\theta_o \frac{\rho V c_P}{hA_s} \left\{ 1 - e^{\frac{-hA_s}{\rho V c_P} \cdot \tau} \right\}$$

$$= hA_s\theta_o \cdot \frac{\tau}{Bi \times Fo} \left\{ 1 - e^{-Bi \times Fo} \right\} \qquad (13)$$

6. Time Constant and Response of Temperature Measuring Instruments

Measurement of temperature by a thermocouple is an important application of the lumped parameter analysis. The response of a thermocouple is defined as the time required for the thermocouple to attain the source temperature.

It is evident from equation (4), that the larger the quantity $\frac{hA_s}{\rho V c_P}$, the faster the exponential term will approach zero or the more rapid will be the response of the temperature measuring device. This can be accomplished either by increasing the value of "h" or by decreasing the wire diameter, density and specific heat. Hence, a very thin wire is recommended for use in thermocouples to ensure a rapid response (especially when the thermocouples are employed for measuring transient temperatures).

From equation (8);

$$\frac{Bi \times Fo}{\tau} = \frac{hA_s}{\rho V c_P}$$
$$\frac{\rho V c_P}{hA_s} = \frac{\tau}{Bi \times Fo}$$

The quantity $\frac{\rho V c_P}{hA_s}$ (which has units of time) is called time constant and is denoted by the symbol τ^*

Thus, $$\frac{\tau}{Bi \times Fo} = \tau^* = \frac{\rho V c_P}{hA_s} = \frac{k}{\alpha h} \cdot \frac{V}{A_s} \qquad (14)$$

$$\left\{\text{Since } \alpha = \frac{k}{\rho c_p}\right\}, \text{And,}$$

$$\frac{\theta}{\theta_o} = \frac{T(t) - T_\infty}{T_o - T_\infty} = e^{-Bi \times Fo} = e^{-(\tau/\tau^*)} \qquad (15)$$

At $\tau = \tau^*$ (one time constant), we have from equation (15),

$$\frac{\theta}{\theta_o} = \frac{T(t) - T_\infty}{T_o - T_\infty} = e^{-1} = 0.368 \qquad (16)$$

Thus, τ^* is the time required for the temperature change to reach 36.8% of its final value in response to a step change in temperature. In other words, temperature difference would be reduced by 63.2%. The time required by a thermocouple to reach its 63.2% of the value of initial temperature difference is called its sensitivity.

Depending upon the type of fluid used the response times for different sizes of thermocouple wires usually vary between 0.04 to 2.5 seconds [1], [3] and [6].

7. Transient Heat Conduction in Solids with Finite Conduction and Convective Resistances $[\mathbf{0} < \boldsymbol{Bi} < \mathbf{100}]$

As shown in Figure 4 below, consider the heating and cooling of a plane wall having a thickness of 2L and extending to infinity in y and z directions. Let us assume that the wall, initially, is at uniform temperature T_o and both the surfaces $(x = \pm L)$ are suddenly exposed to and maintained at the ambient (i.e. surrounding) temperature T_∞. The governing differential equation is:

$$\frac{d^2t}{dx^2} = \frac{1}{\alpha}\frac{dt}{d\tau} \qquad (17)$$

The boundary conditions are:

(i) At $\tau = 0$, $T(t) = T_0$

(ii) At $\tau = 0$, $\frac{dT(t)}{dx} = 0$

(iii) At $x = \pm L$; $\mathrm{kA}\frac{d\mathrm{T(t)}}{dx} = \mathrm{hA(T(t)} - T_\infty)$

(The conduction heat transfer equals convective heat transfer at the wall surface).

The solutions obtained after rigorous mathematical analysis indicate that:

$$\frac{T(t) - T_\infty}{T_o - T_\infty} = f\left[\frac{x}{L}, \frac{hl}{k}, \frac{\alpha\tau}{l^2}\right] \qquad (18)$$

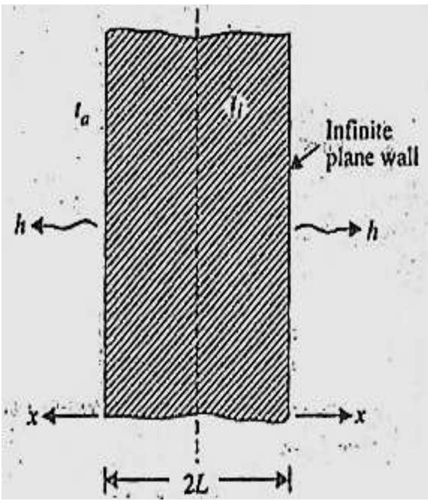

Figure 4: Transient Heat Conduction in An Infinite Plane Wall

From equation (18), it is evident that when conduction resistance is not negligible, the temperature history becomes a function of Biot numbers $\left\{\frac{hl}{k}\right\}$, Fourier number $\left\{\frac{\alpha\tau}{l^2}\right\}$ and the dimensionless parameter $\left\{\frac{x}{L}\right\}$ which indicates the location of point within the plate where temperature is to be obtained. The dimensionless parameter $\left\{\frac{x}{L}\right\}$ is replaced by $\left\{\frac{r}{R}\right\}$ in case of cylinders and spheres.

For the equation (18) graphical charts have been prepared in a variety of forms. In the Figures from 5 to 7 the Heisler charts are shown which depict the dimensionless temperature $\left[\frac{T_c-T_\infty}{T_o-T_\infty}\right]$ versus Fo (Fourier number) for various values of $(\frac{1}{Bi})$ for solids of different geometrical shapes such as plates, cylinders and spheres. These charts provide the temperature history of the solid at its mid – plane $(x = 0)$ and the temperatures at other locations are worked out by multiplying the mid – plane temperature by correction factors read from charts given in figures 8 to 10. The following relationship is used:

$$\frac{\theta}{\theta_o} = \frac{T(t) - T_\infty}{T_o - T_\infty} = \left[\frac{T_c - T_\infty}{T_o - T_\infty}\right] \times \left[\frac{T(t) - T_\infty}{T_c - T_\infty}\right]$$

The values Bi (Biot number) and Fo (Fourier number), as used in Heisler charts, are evaluated on the basis of a characteristic parameter s which is the semi – thickness in the case of plates and the surface radius in case of cylinders and spheres.

When both conduction and convection resistances are almost of equal importance the Heister charts are extensively used to determine the temperature distribution [1], [3] and [6].

Figures 5, 6, 7, 8, 9, and 10 present various Heisler Charts related to temperature history in different geometries. Specifically, Figure 5 illustrates the Heisler Chart for temperature history at the center of a plate with a thickness of 2L, corresponding to (x/L) = 0. Figure 6 depicts the temperature history in a cylinder, while Figure 7 focuses on temperature history in a sphere. Additionally, Figure 8 presents the Heisler Position – Correction Factor Chart for temperature history in a plate, Figure 9 covers the correction factor chart for a cylinder, and Figure 10 concludes with the correction factor chart for a sphere.

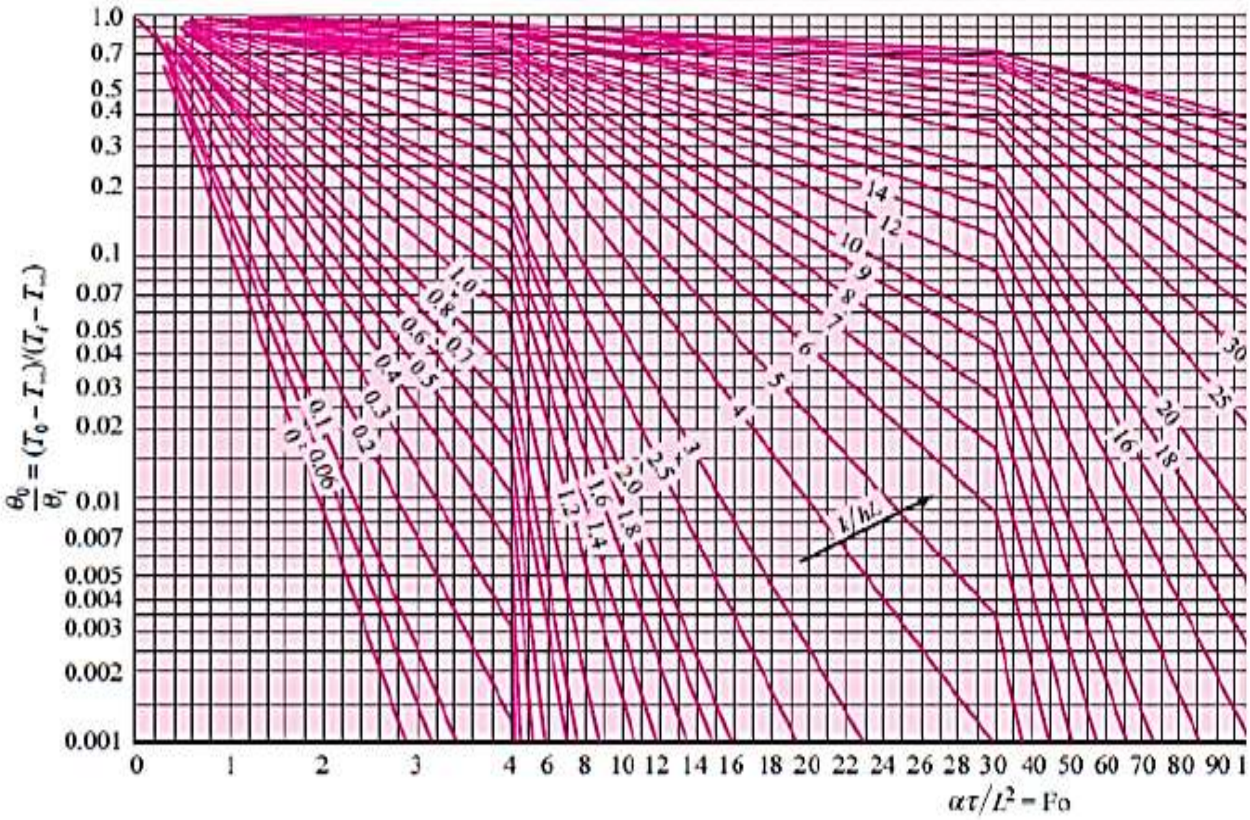

Figure 5: Heisler Chart for Temperature History at the Center of a Plate of Thickness 2L or (x/L) = 0

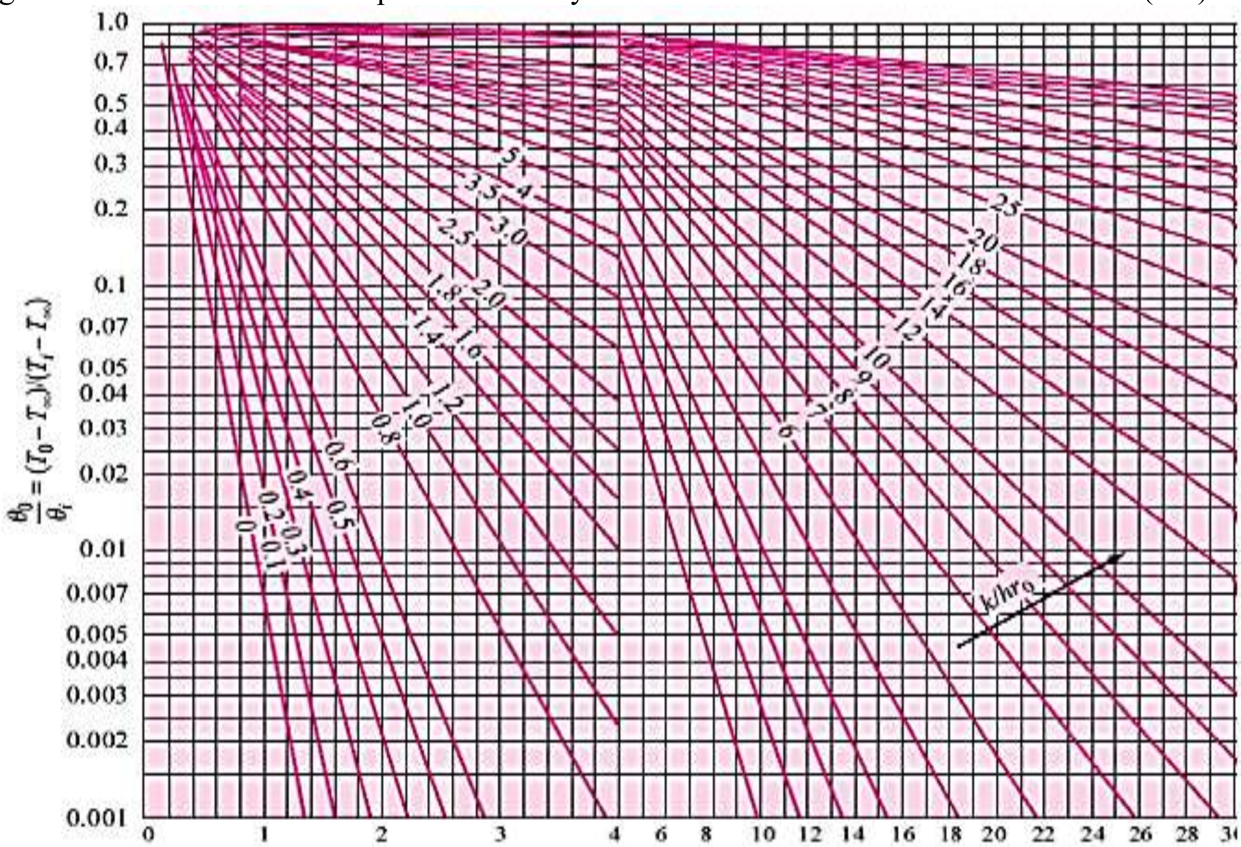

Figure 6: Heisler Chart for Temperature History in a Cylinder

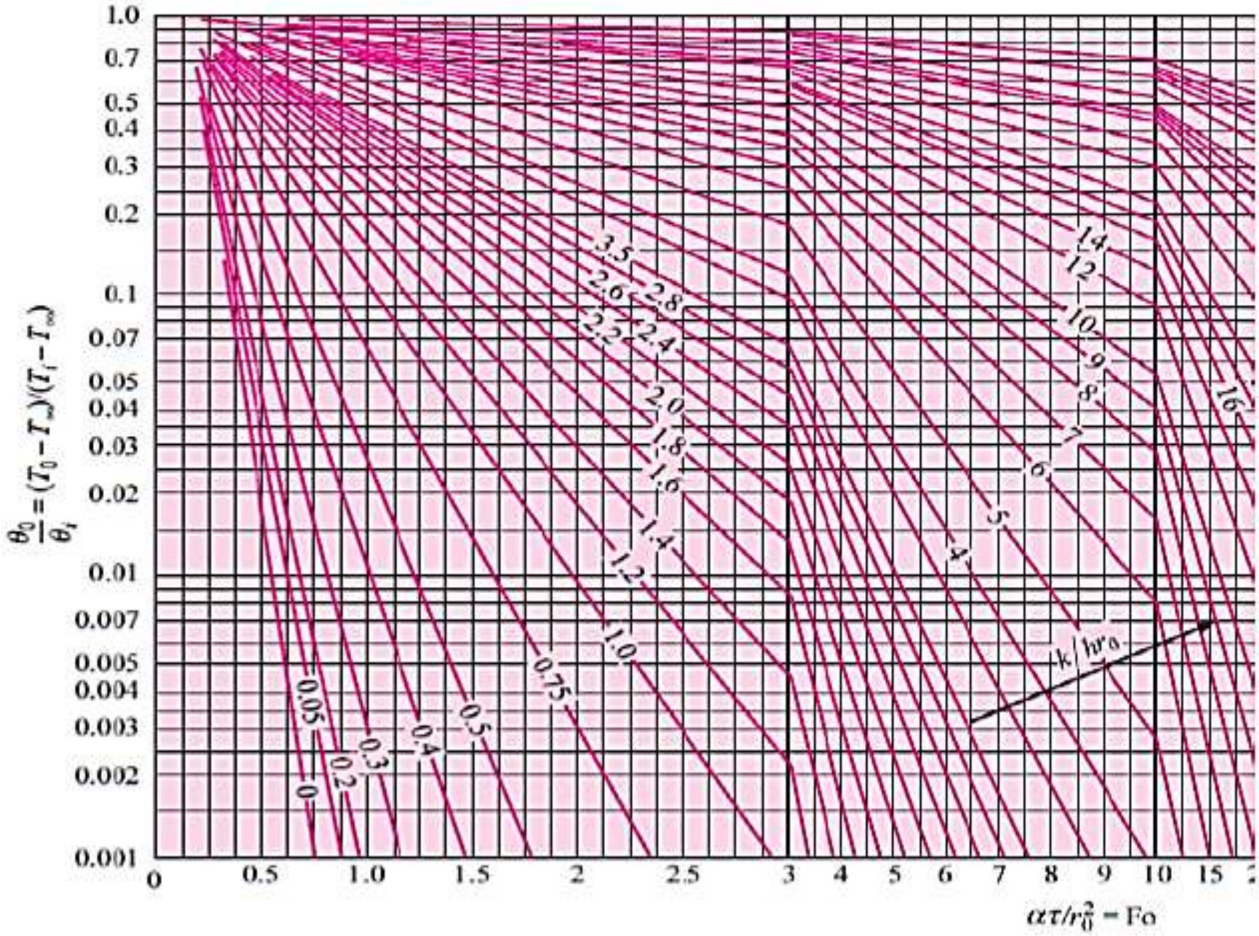

Figure 7: Heisler Chart for Temperature History in a Sphere

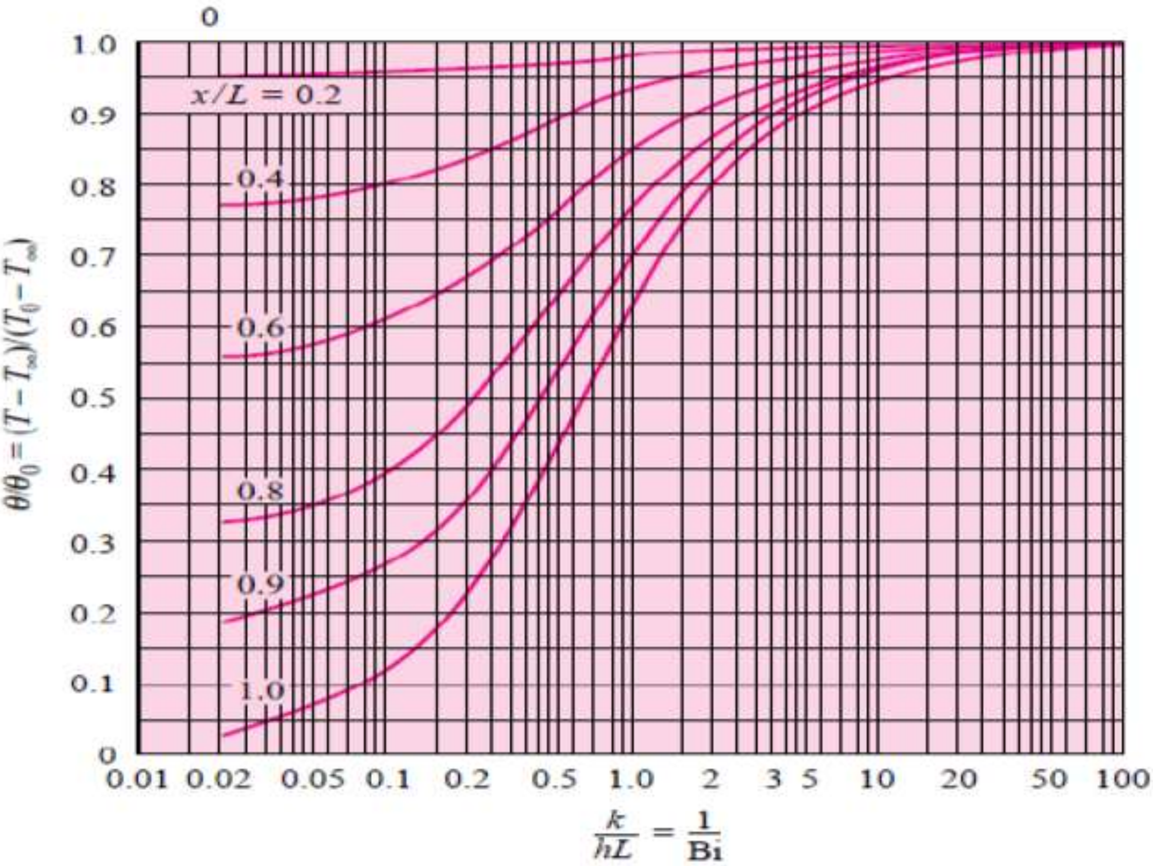

Figure 8: Heisler Position – Correction Factor Chart for Temperature History in Plate

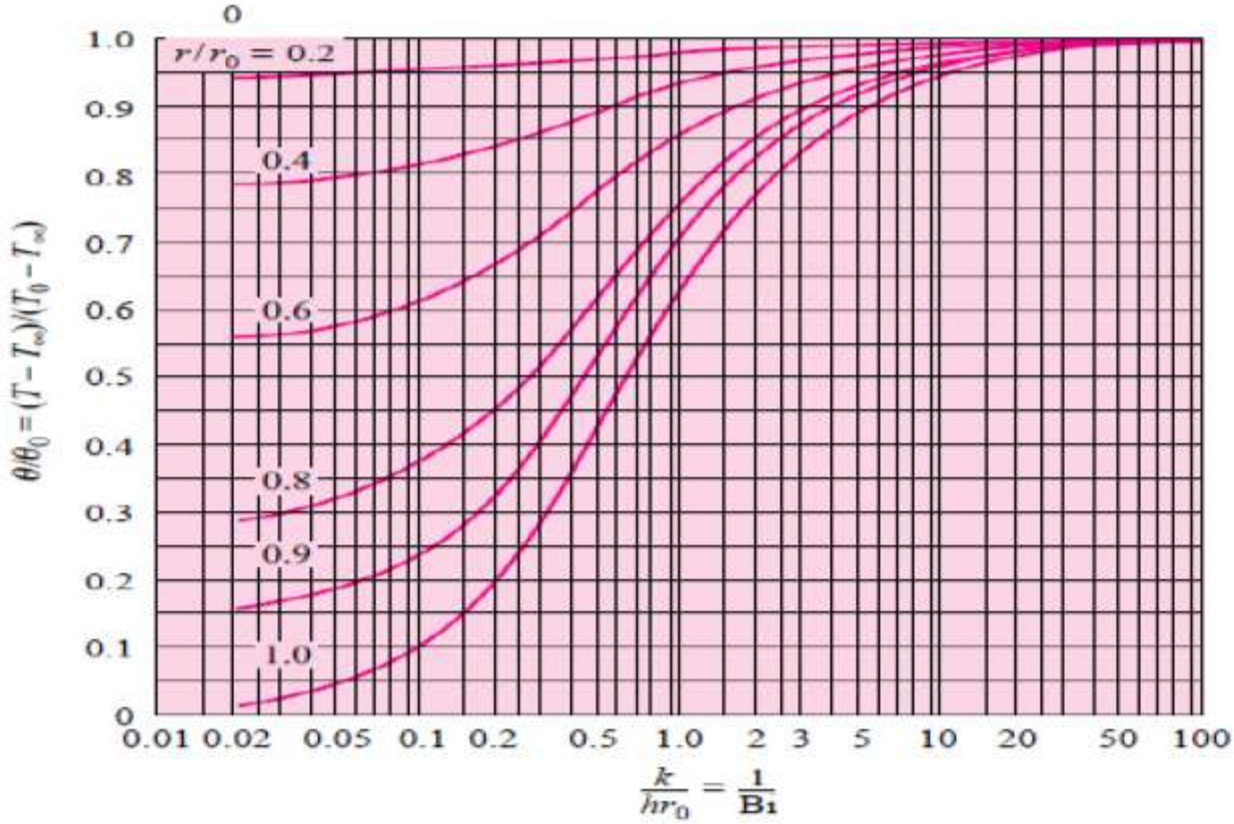

Figure 9 Heisler Position – Correction Factor Chart for Temperature History in Cylinder

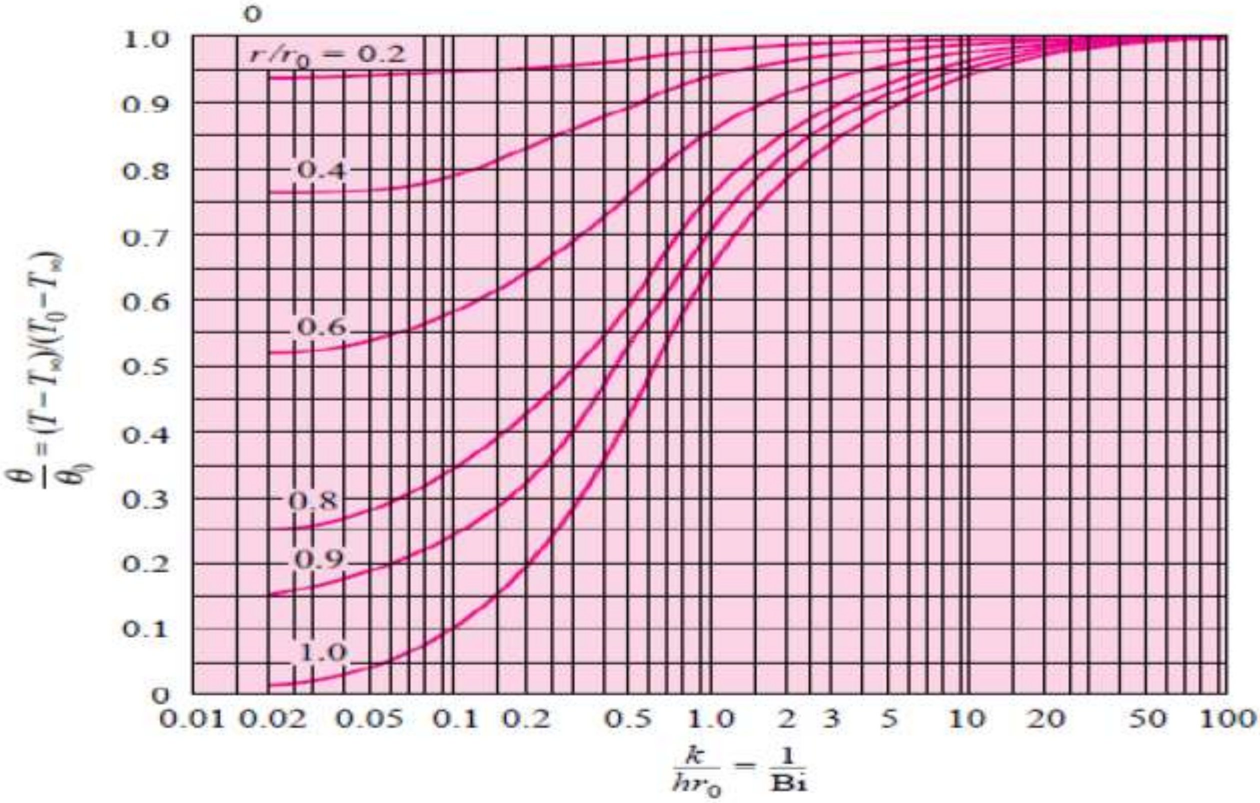

Figure 10 Heisler Position – Correction Factor Chart for Temperature History in Sphere

8. Transient Heat conduction in semi – infinite solids $[H\ or\ Bi\ \to\ \infty]$

8.1 Introduction

A solid which extends itself infinitely in all directions of space is termed as an infinite solid. If an infinite solid is split in the middle by a plane, each half is known as semi – infinite solid. In a semi – infinite body, at any instant of time, there is always a point where the effect of heating (or cooling) at one of its boundaries is not felt at all. At the point the temperature remains unaltered. The transient temperature change in a plane of infinitely thick wall is similar to that of a semi – infinite body until enough time has passed for the surface temperature effect to penetrate through it.

As shown in Figure 11 below, consider a semi – infinite plate, a plate bounded by a plane $x = 0$ and extending to infinity in the $(+ve)$ x – direction. The entire body is initially at uniform temperature T_o including the surface at $x = 0$. The surface temperature at $x = 0$ is suddenly raised to T_∞ for all times greater than $\tau = 0$. The governing equation is:

$$\frac{d^2t}{dx^2} = \frac{1}{\alpha}\frac{dt}{d\tau} \qquad (19)$$

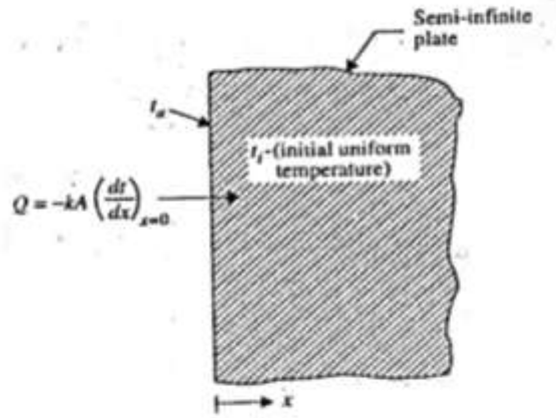

Figure 11: Transition Heat Flow in a Semi – Infinite Plate

The boundary conditions are:

(i) $T(x, 0) = T_o$;

(ii) $T(0, \tau) = T_\infty\ for\ \tau > 0$;

(iii) $T(\infty, \tau) = T_o\ for\ \tau > 0$;

The solution of the above differential equation, with these boundary conditions, for temperature distribution at any time τ at a plane parallel to and at a distance x from the surface is given by:

$$\frac{T(x,\tau) - T_\infty}{T_o - T_\infty} = \mathrm{erf}(z) = erf\left[\frac{x}{2\sqrt{\alpha\tau}}\right] \qquad (20)$$

Where $z = \frac{x}{2\sqrt{\alpha\tau}}$ is known as Gaussian error function and is defined by:

$$erf\left[\frac{x}{2\sqrt{\alpha\tau}}\right] = \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\eta^2}\,d\eta \qquad (21)$$

With $\mathrm{erf}(0) = 0, \mathrm{erf}(\infty) = 1$.

Table 1 shows a few representative values of $\mathrm{erf}(z)$. Suitable values of error functions may be obtained from Figure 12 below.

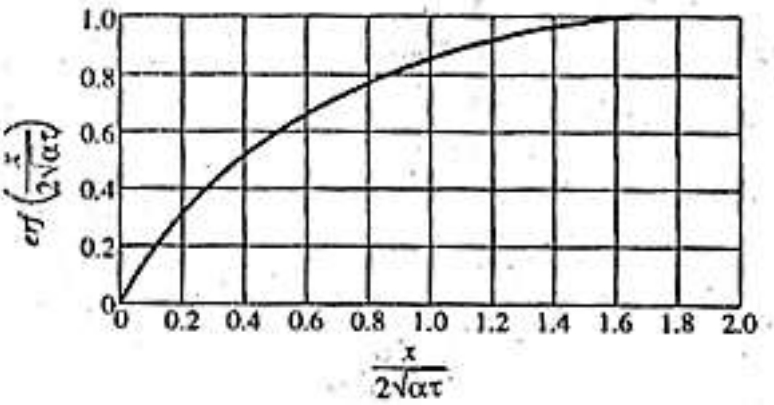

Figure 12: Gauss's Error Integral

Table 1: The Error Function

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\eta^2}d\eta \qquad \text{where } z = \frac{x}{2\sqrt{\alpha\tau}}$$

z	erf(z)	z	erf(z)
0.00	0.0000	0.32	0.3491
0.02	0.0225	0.34	0.3694
0.04	0.0451	0.36	0.3893
0.06	0.0676	0.38	0.4090
0.08	0.0901	0.40	0.4284
0.10	0.1125	0.42	0.4475
0.12	0.1348	0.44	0.4662
0.14	0.1569	0.46	0.4847
0.16	0.1709	0.48	0.5027
0.18	0.2009	0.50	0.5205
0.20	0.2227	0.55	0.5633
0.22	0.2443	0.60	0.6039
0.024	0.2657	0.65	0.6420
0.26	0.2869	0.70	0.6778
0.28	0.3079	0.75	0.7112
0.30	0.3286	0.80	0.7421
0.85	0.7707	1.65	0.9800
0.90	0.7970	1.70	0.9883
0.95	0.8270	1.75	0.9864
1.00	0.8427	1.80	0.9891
1.05	0.8614	1.85	0.9909
1.10	0.8802	1.90	0.9928
1.15	0.8952	1.95	0.9940
1.20	0.9103	2.00	0.9953
1.25	0.9221	2.10	0.9967
1.30	0.9340	2.20	0.9981
1.35	0.9431	2.30	0.9987
1.40	0.9523	2.40	0.9993
1.45	0.9592	2.50	0.9995
1.50	0.9661	2.60	0.9998
1.55	0.9712	2.80	0.9999
1.60	0.9763	3.00	1.0000

By insertion of definition of error function in equation (20), we get

$$T(x,\tau) = T_\infty + (T_o - T_\infty)\frac{2}{\sqrt{\pi}}\int_0^z e^{-\eta^2}\, d\eta$$

On differentiating the above equation, we obtain

$$\frac{\partial T}{\partial x} = \frac{T_o - T_\infty}{\sqrt{\pi\,\alpha\,\tau}}\; e^{[-x^2/(4\,\alpha\,\tau)]}$$

∴ The instantaneous heat flow rate at a given x – location within the semi – infinite body at a specified time is given by:

$$Q_{instantaneous} = -kA(T_o - T_\infty)\frac{e^{\left[-\frac{x^2}{4\,\alpha\,\tau}\right]}}{\sqrt{\pi\,\alpha\,\tau}} \qquad (22)$$

By substituting the gradient $\left[\frac{\partial T}{\partial x}\right]$ in Fourier's law.

The heat flow rate at the surface $(x = 0)$ is given by:

$$Q_{surface} = \frac{-kA(T_o - T_\infty)}{\sqrt{\pi\,\alpha\,\tau}} \qquad (23)$$

∴ The total heat flow rate,

$$Q(t) = \frac{-kA(T_o - T_\infty)}{\sqrt{\pi\,\alpha}}\int_0^\tau \frac{1}{\sqrt{\pi}}d\tau = -kA(T_o - T_\infty)2\sqrt{\frac{\tau}{\pi\alpha}}$$

Or

$$Q(t) = -1.13kA(T_o - T_\infty)\sqrt{\frac{\tau}{\alpha}} \qquad (24)$$

The general criterion for the infinite solution to apply to a body of finite thickness (slab) subjected to one dimensional heat transfer is:

$$\frac{L}{2\sqrt{\alpha\,\tau}} \geq 0.5$$

Where, L = thickness of the body.

The temperature at the center of cylinder or sphere of radius R, under similar conditions of heating or cooling, is given as follows:

$$\frac{T(t) - T_\infty}{T_o - T_\infty} = erf\left[\frac{\alpha\tau}{R^2}\right] \qquad (25)$$

For the cylindrical and spherical surfaces, the values of function $erf\left[\frac{\alpha\tau}{R^2}\right]$ can be obtained from Figure 13 which is shown below [1], [3] and [6].

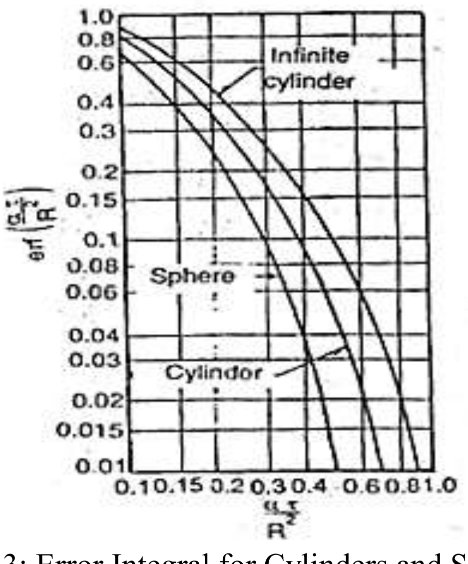

Figure 13: Error Integral for Cylinders and Spheres

8.2 Penetration Depth and Penetration Time

Penetration depth refers to the location of a point where the temperature change is within 1 percent of the change in the surface temperature.

$$i.e.\quad \frac{T(t) - T_\infty}{T_o - T_\infty} = 0.9$$

This corresponds to $\frac{x}{2\sqrt{\alpha\tau}} = 1.8$, from the table for Gaussian error integral.

Thus, the depth (d) to which the temperature perturbation at the surface has penetrated,

$$d = 3.6\sqrt{\alpha\,\tau}$$

Penetration time is the time τ_P taken for a surface penetration to be felt at that depth in the range of 1 percent. It is given by:

$$\frac{d}{2\sqrt{\alpha\,\tau_P}} = 1.8$$

Or

$$\tau_P = \frac{d^2}{13\,\alpha} \qquad (26)$$

9. Systems with Periodic Variation of Surface Temperature

The periodic type of heat flow occurs in cyclic generators, in reciprocating internal combustion engines and in the earth as the result of daily cycle of the sun. These periodic changes, in general, are not simply sinusoidal but rather complex. However, these complex changes can be approximated by a number of sinusoidal components.

Let us consider a thick plane wall (one dimensional case) whose surface temperature alters according to a sine function as shown in Figure 14 below. The surface temperature oscillates about the mean temperature level t_m according to the following relation:

$$\theta_{s,\tau} = \theta_{s,a}\sin(2\pi n\tau)$$

Where:

$\theta_{s,\tau}$ = excess over the mean temperature $(= t_{s,\tau} - t_m)$;

$\theta_{s,a}$ = Amplitude of temperature excess, i.e., the maximum temperature excess at the surface;

n = Frequency of temperature wave.

The temperature excess at any depth x and time τ can be expressed by the following relation:

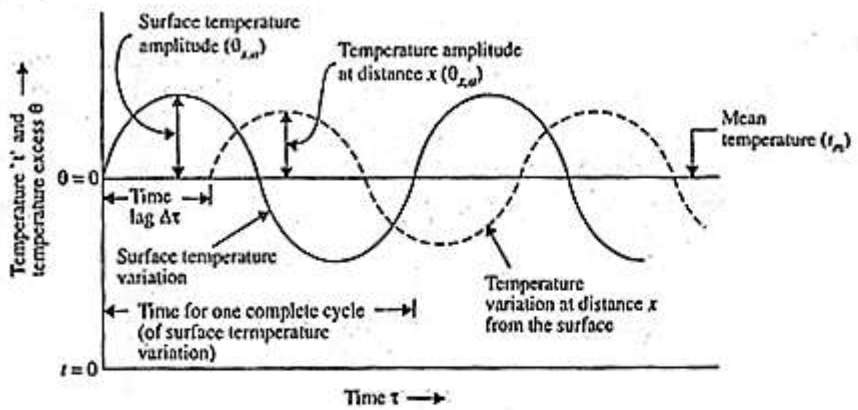

Figure 14: Temperature Curves for Periodic Variation of Surface Temperature

$$\theta_{x,\tau} = \theta_{s,a}\exp\left[-x\sqrt{\pi n/\alpha}\right] sin\left[2\pi n\tau - x\sqrt{\frac{\pi n}{\alpha}}\right] \qquad (27)$$

The temperature excess, at the surface ($x = 0$), becomes zero at $\tau = 0$. But at any depth, $x > 0$, a time $\left[\left[\frac{x}{2}\right]\left[\frac{1}{\sqrt{\alpha\,\pi\,n}}\right]\right]$ would elapse before the temperature excess $\theta_{x,\tau}$ becomes zero. The time interval between the two instant is called the time lag.

$$\text{The time lag} \quad \Delta\tau = \frac{x}{2}\sqrt{\frac{1}{\alpha\,\pi\,n}} \qquad (28)$$

At depth x, the temperature amplitude ($\theta_{x,a}$) is given by:

$$\theta_{x,a} = \theta_{s,a}\exp\left[-x\sqrt{\frac{\pi\,n}{\alpha}}\right] \qquad (29)$$

The above relations indicate the following facts:

1. At any depth, $x > 0$, the amplitude (maximum value) occurs late and is smaller than that at the surface ($x = 0$).
2. The amplitude of temperature oscillation decreases with increasing depth. (Therefore, the amplitude becomes negligibly small at a particular depth inside the solid and consequently a solid thicker than this particular depth is not of any importance as far as variation in temperature is concerned).
3. With increasing value of frequency, time lag and the amplitude reduce.
4. Increase in diffusivity α decreases the time lag but keeps the amplitude large.
5. The amplitude of temperature depends upon depth x as well as the factor $\sqrt{\frac{n}{\alpha}}$. Thus, if $\sqrt{\frac{n}{\alpha}}$ is large, equation (29) holds good for thin solid rods, [1], [3] and [6].

10. Transient Conduction with Given Temperature Distribution

The temperature distribution at some instant of time, in some situations, is known for the one – dimensional transient heat conduction through a solid.

The known temperature distribution may be expressed in the form of polynomial

$t = a - bx + cx^2 + dx^3 - ex^4$ Where a, b, c, d and e are the known coefficients. By using such distribution, the one – dimensional transient heat conduction problem can be solved [1], [3] and [6].

11. Conclusion

Transient conduction, or non-steady-state conduction, describes the temporary changes in temperature within an object due to boundary fluctuations or the introduction of heat sources and sinks. Initially, the temperature adjusts towards a new equilibrium, after which heat flow stabilizes and balances in and out of the system, though steady-state conduction may still occur.

For example, when an automobile engine starts, it undergoes transient conduction until reaching a stable temperature, whereas a hot copper ball in cold oil gradually cools without achieving steady-state conditions.

Analyzing non-steady-state conduction is generally more complex than steady-state; simpler shapes may allow for exact solutions, while complex geometries often require numerical methods or approximations like Heisler Charts. High-conductivity regions can be analyzed using a lumped capacitance model for simplicity. In summary, transient heat transfer involves non-uniform heat energy transfer influenced by varying temperature differences and material properties.

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